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NON-ARCHIMEDEAN HARMONIC ANALYSIS ON GROUPS WITHOUT HAAR MEASURE

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PROEFSCHRIFT

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Aan mijn ouders.

Aan Tonny.

Langs deze weg wil ik allen danken die mij, soms zonder dat zij dit zelf beseften, hebben gestimuleerd dit proefschrift te schrijven, vooral mijn ouders.
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CONTENTS

INTRODUCTION AND SUMMARY

This thesis is meant to be a contribution to the development of a theory on non-Archimedean group algebras. Group algebras like $L^1(G)$ and $M(G)$ appear in a natural way in several problems of classical analysis. Often, one tries to obtain information about the structure of a locally compact group G via its group algebras[for instance via the group of algebra homomorphisms from $L^1(G)$ onto $\mathbb{C}$]. On the other hand,group algebras are frequently used to solve problems in applied mathematics [for instance, with the help of the Fourier transform, a mighty weapon!]. Some references are [E], [G-S], [H-R], [P], [R-I], [R-II] and [Wa]. In all these works, the scalar field is either $\mathbb{R}$ or $\mathbb{C}$. In this thesis we consider other scalar fields K, namely the non-Archimedean valued ones. Some people working on related subjects are Amice, Diarra, Duponcheel, Escassut, Fresnel, de Mathan, Monna, van der Put, van Rooij, Schikhof, Springer and Woodcock.

Let G be a locally compact group and let H be its component of unity. The topology on K is zerodimensional. Hence, for example K-valued continuous functions on G are constant on cosets of H. It follows that, for instance, information about the structure of G via its non-Archimedean group algebras is often only information about the structure of the zerodimensional group G/H. For this reason, we will restrict ourselves to zerodimensional groups. In fact, the groups we deal with are the zerodimensional, abelian, torsional ones. [For the definition of a torsional group see (1.10). Examples of such groups are abelian compact groups, products of non-Archimedean normed vector spaces, etc.]

Not all zerodimensional compact groups have a K-valued Haar measure [see v.R-302]. For instance, Z_p has no $\mathbb{Q}_p$-valued Haar measure. Hence, convolution of functions on G cannot always be defined. It follows that there is often no analogue of the classical $L^1(G)$. For this reason, we will mainly concern ourselves with the non-Archimedean analogue of the classical group algebra M(G).

In Chapter I, we gather the material needed in Chapter II - Chapter VI. Most of this material is also treated in [v.R.].

In Chapter II, we prove that each [abelian] torsional group G is in a natural way a direct sum of a p-primary group G_1 and a p-free group G_2. Compact p-free groups have a Haar measure. They are studied for example by v.Rooij and by Schikhof.

We show that information about the group algebras $M(G_1)$ and $M(G_2)$ gives us information about the group algebra M(G). Finally, we show that there are two types of [abelian] p-primary groups, namely groups of type Z_p and groups of type C, in the sense that each [abelian] compact group is a semidirect product of such groups.

The main theorems of Chapter II are (2.1), (2.4), (2.9), (2.12), (2.13), (2.14), (2.19), (2.21).

In Chapter III we consider groups of type Z_p. Let G_1 and G_2 be torsional groups with G_1 of type Z_p. Then $M(G_1 \times G_2)$ is a Banach module over $M(G_1)$ such that $||\mu * \nu|| = ||\mu||\ ||\nu||$ for each $\mu \in M(G_1)$ and $\nu \in M(G_1 \times G_2)$. This implies that the norm in M(G) is multiplicative in the case that G is of type Z_p. Conversely, we prove that multiplicativity of the norm in M(G) implies that G is of type Z_p. For G of type Z_p we determine homomorphisms from $M(Z_p)$ onto K. Finally, we give criteria for invertibility of a measure μ in terms of its Fourier transform $\hat{\mu}$. This chapter is an extension of the work of v.d.Put [P-I]. The main theorems are (3.1), (3.5), (3.11),(3.19),(3,22) and (3.23).

In Chapter IV, p-primary groups are studied in the case that K is of characteristic zero. In the classical case, idempotent measures are a crucial link in the description of homomorphisms between group algebras [see R.I.-Chapter IV]. We conjecture that each idempotent K-valued measure on a p-primary group G has finite support. This conjecture is proved in the case that G is of type Z_p, and in the case that G is isomorphic to $(C_p)^I$ for some index set I. Further, we shall concern ourselves in Chapter IV with Fourier transforms of groups of type C. It is proved that for a group of type C and $\mu \in M(G)$ in general $\hat{\mu} \notin c_\infty(G^*)$. The main theorems are (4.1), (4.15), (4.16), (4.18) and (4.19).

In Chapter V we deal with p-primary groups in the case that K has finite characteristic. For this case we prove the conjecture posed in (4.1). We give criteria for invertibility of a measure. Further we show that the study of finite dimensional continuous representations for groups of type C is, in fact, the study of finite dimensional shift invariant subspaces of C(G). The main of this chapter are (5.2), (5.7), (5.11), (5.14), (5.16) and (5.19).

In Chapter VI, we determine the finite dimensional continuous representations of a group isomorphic to $(C_2)^I$, for some index set I, in the case that the characteristic of K is 2. To achieve this aim, we determine the finite dimensional shift invariant subspaces V of $C(C_2^I)$ [see (5.19)]. We show that for each finite dimensional shift invariant subspace V of $C(C_2^I)$ there is a family $a_1,\dots,a_n$ of additive homomorphisms such that $V \subset [\![\{\bar{s} * \prod_{i=1}^{n} a_i \mid s \in G\}]\!]$. The main theorems are (6.9), (6.26), (6.28) and (6.30).

DEFINITIONS AND CONVENTIONS

In this thesis K is a complete non-Archimedean valued field whose valuation $|.|$ is non-trivial. The unit element of K is 1. The residue class field of K is k. The characteristic of k is p. We always assume that $p \neq 0$. In the case that characteristic K is zero, we assume that $|p.1| = \frac{1}{p}$. Vector spaces [Banach spaces, etc.] are all vector spaces [Banach spaces, etc.] over K.

Within the context of this thesis, we shall only be considering abelian groups.

NOTATIONS

When a word is underlined, the sentence it appears in is understood to define the meaning of the word. In formulas of the type $A \underset{D}{=} B$, it is understood that, by definition, A is the same as B.

The K-valued characteristic function of a set X is denoted by $\zeta(X)$. The cardinality of a set A is denoted by #A. If A and B are sets then $A \setminus B \underset{D}{=} \{x \mid x \in A, x \notin B\}$. Let A be a subset of a topological space B. Then the closure of A is denoted by clo A.

Let I be an index set and $(X_i)_{i \in I}$ a family of topological spaces. On the cartesian product, $\underset{i \in I}{\pi} X_i$, we take the weakest topology for which every coordinate function is continuous.

For a topological group G, $[\![g_1,\dots,g_n]\!]$ is the closure of the subgroup of G generated by the elements $g_1,\dots,g_n \in G$.

For a vectorspace E and $e_1,\dots,e_n \in E$, $[\![e_1,\dots,e_n]\!]$ denotes the linear span of the vectors $e_1,\dots,e_n$.

Let J be an index set and let for each $j \in J$, E_j be a Banach space. Then by $\underset{j \in J}{\times} E_j$ we denote the set of all elements a of $\underset{j \in J}{\pi} E_j$ for which the set $\{||a_j|| \mid j \in J\}$ is bounded. This $\underset{j \in J}{\times} E_j$ is normed under $||a|| \underset{D}{=} \sup\{||a_j|| \mid j \in J\}$. The elements a of $\underset{j \in J}{\pi} E_j$ for which for every $\varepsilon > 0$, the set $\{j \in J \mid ||a_j|| \geq \varepsilon\}$ is finite is a linear subspace of $\underset{j \in J}{\times} E_j$, denoted by $\underset{j \in J}{\oplus} E_j$. Then $\underset{j \in J}{\times} E_j$ and $\underset{j \in J}{\oplus} E_j$ are Banach spaces. Special cases are $l^\infty(J) \underset{D}{=} \underset{j \in J}{\times} K$ and $c_\infty(J) \underset{D}{=} \underset{j \in J}{\oplus} K$.

Let G be a topological group. Then $\hat{G}$ denotes the group of all continuous complex valued characters on G. G^* denotes the group of all continuous K-valued characters on G.

For a subgroup H of G [or G^*], Ann H $\underset{D}{=} \{\gamma \mid \gamma \in G^*, \gamma(x) = 1 \text{ all } x \in H\}$ [or $\{g \mid g \in G, \gamma(g) = 1 \text{ all } \gamma \in H\}$]. We put a uniform structure $\mathcal{U}$ on G^* as follows: $U \in \mathcal{U}$ if there is a compact subgroup H of G such that $\{(\gamma_1,\gamma_2) \mid (\gamma_1,\gamma_2) \in G^* \times G^*, \gamma_1 = \gamma_2 \text{ on } H\} \subset U$. Let E be a Banachspace. Then $BUC(G^* \to E)$ denotes the set of all bounded functions from G^* into E, that are uniformly continuous.

The symbols $\mathbb{Q}_p$ and $\mathbb{F}_p$ denote the field of p-adic numbers and the field consisting of p elements respectively.

Z_p, C_{p^n} and $Z(p^\infty)$ denote the group of p-adic integers, the cyclic group of p^n elements and the discrete group of all [complex] p^{th}-roots of unity, respectively. $\Delta_1(0) \underset{D}{=} \{x \mid x \in K, |x| < 1\}$.

The symbol $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{C}$ stands for the complex numbers.

Terms and notions concerning non-Archimedean Functional Analysis are borrowed from the book of A.van Rooij [v.R.].

The end of proofs, etc. is indicated by the symbol □.

CHAPTER I

PRELIMINARIES

§1. Measures

(1.1) DEFINITION. A zerodimensional topological space is a topological space admitting a basis of the topology consisting of sets which are clopen [i.e. of sets that are both open and closed]. □

Let X be a zerodimensional topological space and let $\mathcal{R}$ denote the ring of clopen sets of X.

(1.2) DEFINITION. A [tight] measure on X is a map $\mu : \mathcal{R} \to K$ with the properties

(i) μ is additive.

(ii) The set $\{\mu(B) \mid B \in \mathcal{R}\}$ is bounded.

(iii) For each $\varepsilon > 0$ there is a compact subset Y of X such that $|\mu(A)| < \varepsilon$ for each $A \in \mathcal{R}$ with $A \cap Y = \emptyset$. □

(1.3) REMARK. In [v.R.] a more general definition of measure is given. However, in our context it can be shown that both definitions coincide. □

M(X) denotes the collection of all measures on X. It is obvious that M(X) is a Banachspace under $||\mu|| = \sup_{A\in\mathcal{R}} |\mu(A)|$ $[\mu \in M(X)]$. For $x \in X$,

$\bar{x} \in M(X)$ denotes the evaluation in x [i.e. $\bar{x}(A) = (\zeta(A))(x)$ for each $A \in \mathcal{R}$]. For a finite subset $H \subset X$, $\bar{H} \in M(X)$ is defined by

$$\bar{H} \underset{D}{=} \Sigma\{\bar{x} \mid x \in H\}.$$

Let Y be a closed subspace of X. With the restriction topology, Y is itself a zerodimensional topological space. Each element of M(Y) defines in a canonical way an element of M(X). Therefore,

M(Y) <u>will be seen as a closed linear subspace of</u> M(X).

BC(X) denotes the Banach space of the bounded continuous function from X to K, normed under $||f|| = \sup_{x \in X} |f(x)|$. For compact X, BC(X) is indicated by C(X).

The characteristic functions of the elements of $\mathcal{R}$ generate a linear subspace $\mathcal{T}$ of BC(X). For $f \in \mathcal{T}$ one defines the integral $\int f d\mu$ in the obvious way, such that $\int \zeta(A) d\mu = \mu(A)$ for all $A \in \mathcal{R}$. It turns out that the functional $f \to \int f d\mu$ restricted to bounded subsets of $\mathcal{T}$, is continuous relative to uniform convergence on compact sets. This observation enables one to define $\int f d\mu$ for $f \in BC(X)$ by continuity. Then

$$|\int f d\mu| \leq ||f||_\infty \, ||\mu|| \quad [f \in BC(X)].$$

For details we refer the reader to [v.R.].

Let BC(X)' denote the dual space of BC(X) [i.e. the space of all continuous linear maps from BC(X) into K]. We norm BC(X)' by

$$||\Phi|| \underset{D}{=} \sup_{||f|| \leq 1} |\Phi(f)|.$$

Define $T : M(X) \to BC(X)'$ by

$$(T\mu)(f) = \int f d\mu \quad [\mu \in M(X),\ f \in BC(X)].$$

Then

(1.4) THEOREM. T is a linear isometry. For compact X, T is surjective.

Proof. See [v.R.-(7.18)]. □

From now on $\mu \in M(X)$ and $T\mu \in BC(X)'$ will be identified.

(1.5) THEOREM. For each $\mu \in M(X)$ and $\varepsilon > 0$ there is a compact subset Y of X together with $\nu \in M(Y)$ such that $||\mu - \nu|| < \varepsilon$.

Proof. Use (1.4) and [v.R.-(5.24)]. □

We can formulate (1.6) in another way. To do this, we first have to introduce another notion.

(1.6) DEFINITION. Let $\mu \in M(X)$. Then the support of μ, denoted by supp μ, is the smallest closed set Y of X for which $\mu(A) = 0$ as soon as $A \in \mathcal{R}$ with $A \cap Y = \emptyset$ [it is well known that such a smallest set Y exists]. □

(1.7) THEOREM. $M(X)$ is the closure of the subset of $M(X)$ consisting of all measures with compact support. □

§2. Zerodimensional groups.

(1.8) DEFINITION. A topological group is a group G provided with a topology which is Hausdorff and for which the map $(g,h) \to g\text{-}h$ $[g,h \in G]$ is continuous. □

The following theorem will often be used in this thesis.

(1.9) THEOREM. Let G be a compact topological group and let X be a clopen subset of G. Then there is an open subgroup H of G such that X is a finite union of cosets of H.

Proof. See [v.R.-(8.2)]. □

In this thesis we will mainly study torsional groups.

(1.10) DEFINITION. A torsional group is a topological group for which each compact subset is contained in a compact subgroup and for which there is a neighbourhood base of zero consisting of open subgroups. □

Proofs of the following theorens concerning torsional groups are left to the reader.

(1.11) THEOREM. Let H be a closed subgroup of a torsional group G. Then H with the relative topology is a torsional group. □

(1.12) THEOREM. Let H be a closed subgroup of a torsional group G. Then G/H with the quotient topology is a torsional group. □

(1.13) THEOREM. Let I be an index set and let $(G_i)_{i\in I}$ be a family of torsional groups. Then $\prod_{i\in I} G_i$ is a torsional group. □

For a topological group G and $\mu, \nu \in M(G)$ one can show that there is an element $\mu * \nu \in M(G)$, called the convolution of μ and ν, such that

$$(\mu * \nu)(f) = \int\int f(x+y)d\mu(x)d\nu(y) \quad \text{for each} \quad f \in BC(G).$$

(1.14) THEOREM. M(G) is a Banachalgebra under convolution.

Proof. Elementary. □

For $f \in BC(G)$ and $\mu \in M(G)$ we define $\mu * f \in BC(G)$ by

$$(\mu * f)(x) = \int_D f(x+y)d\mu(y) \quad [x \in G].$$

(1.15) THEOREM. $||\mu * f|| \leq ||\mu|| \; ||f||$.

Proof. Elementary. □

(1.16) DEFINITION. Let G be a compact group. Then $\mu \in M(G)$ is a *Haar measure* if $\mu \neq 0$ and if $\mu(x+U) = \mu(U)$ for each $x \in G$ and each clopen subset U of G. μ is *normalized* if $\mu(G) = 1$. □

(1.17) THEOREM. Let G be a compact group and let $\mu, \mu' \in M(G)$ be Haar measures. Then there is $\lambda \in K$ with $\mu = \lambda . \mu'$.

Proof. See [v.R.-(8.4)]. □

Again we introduce some new notions.

(1.18) DEFINITION. A topological group G is a p-*primary* group if $\lim_{n\to\infty} p^n g = 0$ for each $g \in G$. □

(1.19) DEFINITION. A compact group G is a p-*free* group if for each open subgroup H of G the index $[G \cdot H]$ is not divisible by p. A torsional group G is p-*free* if each compact subgroup is p-free. □

(1.20) THEOREM. (a) Let G be a compact p-free group. Then G has a unique normalized Haar measure.

(b) Let G be a compact p-primary group. Then G has a Haar measure if and only if G is finite.

Proof. See [v.R.-(8.4)]. □

Let G be compact and p-free. Let m be the normalized Haar measure on G. Define <u>convolution</u> on C(G) by

$$(f * g)(x) \underset{D}{=} \int f(x+y)g(-y)\,dm(y) \qquad [f,g \in C(G)].$$

Then

(1.21) THEOREM. C(G) with convolution is a Banach algebra.

Proof. Elementary. □

For a compact group that is p-free we define the map $S : C(G) \to M(G)$ by

$$Sf \underset{D}{=} f.m \quad [\text{i.e. } (Sf)(g) = \int f(x)g(x)\,dm(x)].$$

(1.22) THEOREM. The map S is a Banach algebra isomorphism of the convolution algebra C(G) onto a closed ideal of M(G).

Proof. See [v.R.-(8.4)]. □

<u>From now on for a p-free compact group</u> G, <u>we will identify</u> C(G) <u>with a closed ideal of</u> M(G).

§3. <u>Miscellaneous facts</u>.

(A) <u>Tensor products</u>. Let X and Y be zerodimensional Hausdorff spaces. For $f \in BC(X)$ and $g \in BC(Y)$ define $f \otimes g \in BC(X \times Y)$ by

$$(f \otimes g)(x,y) \underset{D}{=} f(x).g(y).$$

Then,

(1.23) THEOREM. Let X and Y be compact. Then $C(X\times Y)$ is the closure of the linear span of the functions $f \otimes g$ $[f \in C(X),\ g \in C(Y)]$.

Proof. See [v.R.- p. 131]. $\square$

(1.24) COROLLARY. Let X and Y be zerodimensional Hausdorff spaces and let $\mu \in M(X\times Y)$. Then μ is completely determined by the values of $\mu(f \otimes g)$ $[f \in BC(X),\ g \in BC(Y)]$.

Proof. Use (1.5) and (1.23). $\square$

(1.25) COROLLARY. Let X and Y be compact, zerodimensional and Hausdorff. Let $(f_i)_{i\in I}$ be an orthonormal base of $C(X)$ and $(g_j)_{j\in J}$ an orthonormal base of $C(Y)$. Then $(f_i \otimes g_j)_{(i,j)\in I\times J}$ is an orthonormal base of $C(X\times Y)$.

Proof. Use (1.23). $\square$

(1.26) THEOREM. Let X and Y be compact, zerodimensional and Hausdorff. Let $L(C(X),M(Y))$ denote the Banachspace of all continuous linear maps from $C(X)$ into $M(Y)$, normed by $||A|| \underset{D}{=} \sup\limits_{||f||\leq 1} ||A(f)||$.
Then the map

$$R : M(X\times Y) \to L(C(X),\ M(Y))$$

defined by

$$((R\mu)(f))(g) = \mu(f \otimes g) \quad [\mu \in M(X\times Y),\ f \in C(X),\ g \in C(Y)]$$

is a surjective linear isometry.

Proof. See [v.R.-(4.27)]. $\square$

(B) The Banach algebra $K||X_1,\ldots,X_n||$.

(1.27) DEFINITION. Let $n \in \mathbb{N}$. The formal power series $\sum_{m(1),\ldots,m(n)\in\mathbb{N}\cup\{0\}} a_{m(1),\ldots,m(n)} X_1^{m(1)} \cdot\ldots\cdot X_n^{m(n)}$ for which the set of coefficients $\{a_{m(1),\ldots,m(n)} \mid m(1),\ldots,m(n) \in \mathbb{N}\cup\{0\}\}$ is bounded form a Banach algebra under

$$\Big\| \sum_{m(1),\ldots,m(n)\in\mathbb{N}\cup\{0\}} a_{m(1),\ldots,m(n)} X_1^{m(1)} \cdot \ldots X_n^{m(n)} \Big\| \underset{D}{=}$$

$$\sup\{|a_{m(1),\ldots,m(n)}| \mid m(1),\ldots,m(n) \in \mathbb{N}\cup\{0\}\}.$$

This Banach algebra is denoted by $K\|X_1,\ldots,X_n\|$. □

(1.28) THEOREM. Let $\Phi : K\|X_1,\ldots,X_n\| \to K$ be a homomorphism. Then there are $\lambda_1,\ldots,\lambda_n \in K$, $|\lambda_i| < 1$, such that $\Phi(f) = f(\lambda_1,\ldots,\lambda_n)$ for each $f \in K\|X_1,\ldots,X_n\|$. Conversely, each $\lambda_1,\ldots,\lambda_n \in K$ with $|\lambda_i| < 1$ determines a homomorphism in this way.

<u>Proof</u>. Compare [v.R.-p.235-p.236]. □

(1.29) THEOREM. Let K be algebraically closed and let $f \in K\|X_1,\ldots,X_n\|$, $f \neq 0$, $f = \sum_{m(1),\ldots,m(n)\in\mathbb{N}\cup\{0\}} a_{m(1),\ldots,m(n)} X_1^{m(1)} \ldots X_n^{m(n)}$.

Then the following statements are equivalent:

(i) f is invertible;

(ii) $\|f\| = |a_{0,\ldots,0}|$;

(iii) $f(\lambda,\ldots,\lambda_n) \neq 0$ for each $\lambda_1,\ldots,\lambda_n \in K$, $|\lambda_i| < 1$.

<u>Proof</u>. (i) ⇒ (iii) is trivial.

(iii) ⇒ (ii).

(a) <u>The case that</u> $n = 1$.

Suppose that $\|f\| > |a_0|$. Then we can choose $k \in \mathbb{N}$ and $\pi \in K$,

$|\pi| < 1$, such that $|\pi^k a_k| > |a_0|$. Define f_π by $f_\pi \underset{D}{=} \Sigma\, a_n \pi^n X^n$.
From [v.R.-(6.37)] and [v.R.-(6.39)] we deduce that there is an $\alpha \in K$ with $|\alpha| \leq 1$ such that $f_\pi(\alpha) = 0$. Then clearly $f(\pi\alpha) = 0$, while $|\pi\alpha| < 1$.

(b) The case that $n > 1$. Let $f_0, f_1, \ldots \in K||X_1,\ldots,X_{n-1}||$ be such that $f = \Sigma\, f_m X_n^m$.

(b.1) Let $||f|| = ||f_0||$. Then by using an induction argument it follows that we can choose $\alpha_1,\ldots,\alpha_{n-1} \in K$, $|\alpha_i| < 1$, with $f_0(\alpha_1,\ldots,\alpha_{n-1}) = 0$. Hence, $f(\alpha_1,\ldots,\alpha_{n-1},0) = 0$.

(b.2) Let $||f_k|| > ||f_0||$ for some $k \in \mathbb{N}$. Then there exist $\alpha_1,\ldots,\alpha_{n-1} \in K$, $|\alpha_i| < 1$, with $|f_k(\alpha_1,\ldots,\alpha_{n-1})| > ||f_0||$ [see [v.R.-(6.42)] Then by using (a), it follows that there is an $\alpha_n \in K$, $|\alpha_n| < 1$, with $f(\alpha_1,\ldots,\alpha_{n-1},\alpha_n) = 0$.

(ii) $\Rightarrow$ (i). Inductively we can construct $b_{l(1),\ldots,l(n)} \in K$ such that

1) $\sum_{m(i)\leq l(i)} a_{m(1),\ldots,m(n)}\, b_{l(1)-m(1),\ldots,l(n)-m(n)} = 0$ if $(l(1),\ldots,l(n)) \neq (0,\ldots,0)$ and

$\sum_{m(i)\leq l(i)} a_{m(1),\ldots,m(n)}\, b_{l(1)-m(1),\ldots,l(n)-m(n)} = 1$ if $(l(1),\ldots,l(n)) = (0,\ldots,0)$.

2) $|b_{l(1),\ldots,l(n)}| < |a_{0,\ldots,0}|^{-1}$.

Then it is clear that

$$\Sigma\, b_{l(1),\ldots,l(n)} X_1^{l(1)} \cdots X_n^{l(n)} \text{ is the inverse of } f. \quad \square$$

(C) The Banach algebra $l^\infty(I)$.

(1.30) DEFINITION. Let X be a set and let $\mathcal{B}(X)$ denote the set of all subsets of X. Then $\mathcal{M} \subset \mathcal{B}(X)$ is a filter if

(i) $\emptyset \notin \mathcal{M}$.

(ii) If $A_1, \ldots, A_n \in \mathcal{M}$ then $A_1 \cap \ldots \cap A_n \in \mathcal{M}$.

(iii) If $A \in \mathcal{M}$, $B \in \mathcal{B}(X)$ and $A \subset B$ then $B \in \mathcal{M}$.

An ultrafilter is a maximal filter.

An ultrafilter $\mathcal{M}$ is fixed if there is $x \in X$ such that $\{x\} \in \mathcal{M}$. Then $\mathcal{M} = \{A \mid A \in \mathcal{B}(X), x \in A\}$. An ultrafilter that is not fixed is a free ultrafilter. □

(1.31) THEOREM. Let K be locally compact and let I be an index set. Let $\Phi : l^\infty(I) \to K$ be a homomorphism [l^∞ with pointwise multiplication]. Then there is an ultrafilter $\mathcal{M}$ on I such that

$$\Phi(f) = \bigcap_{A \in \mathcal{M}} \mathrm{clo}\{f(i) \mid i \in A\} \text{ for each } f \in l^\infty(I).$$

Conversely, each ultrafilter on I determines a homomorphism in this way.

Proof. See [v.R.-(6.4)]. □

A cardinal number m is said to be measurable if there is a set X with $\#X = m$ together with a non-zero σ-additive measure $\mu : \mathcal{B}(X) \to \{0,1\}$, that is not an evaluation in a point.

It is obvious that $\#\mathbb{N}$ is non-measurable.

For the proof of the following theorem we refer to [G-J-(12.5)].

(1.32) THEOREM. Let X be a set such that $\#X$ is non-measurable. Then,

(i) $\#\mathcal{B}(X)$ is non-measurable.

(ii) If Y is a subset of X then $\#Y$ is non-measurable.

(iii) If Z is a set with $\#Z$ non-measurable,

then $\#(X \times Z)$ is non-measurable. $\square$

It is for reasons of (1.33) that we make the following assumption.

<u>From now on in this thesis we restrict ourselves to sets whose cardinal number is non-measurable</u>.

(1.33) THEOREM. Let K be not locally compact and let I be an index set. Let $\Phi : l^{\infty}(I) \to K$ be a homomorphism. Then there is $i \in I$ such that $\Phi(f) = f(i)$ for each $f \in l^{\infty}(I)$.

<u>Proof</u>. Suppose that $\Phi : l^{\infty}(I) \to K$ is a homomorphism not of the type mentioned in the theorem. Then we prove that necessarily $\#I$ must be measurable, which is a contradiction with the assumption above.

Define $\mu \in \mathcal{B}(I)$ by the formula

$$\mu(U) \underset{D}{=} \Phi(\zeta(U)).$$

Then μ is non-trivial, $\mu : \mathcal{B}(X) \to \{0,1\}$ and μ is not a point measure. What is left is the proof of the fact that μ is σ-additive. Assume for the moment that the following is true.

$$* \begin{cases} \text{Let } V_1, V_2, \ldots \text{ be disjoint subsets of } I \text{ with } \bigcup_{i=1}^{\infty} V_i = I. \text{ Then there is} \\ n \in \mathbb{N} \text{ with } \Phi(\zeta(V_i)) = 0 \text{ when } i \neq n \text{ and } \Phi(\zeta(V_n)) = 1. \end{cases}$$

Now let $U_1, U_2, \ldots$ be disjoint subsets of I. Define U_0 as $U_0 = I \setminus \bigcup_{i=1}^{\infty} U_i$.

It follows from (*) that $\mu(U_n) = 1$ for exactly one $n \in \mathbb{N} \cup \{0\}$. Using this, we easily deduce that

$$\sum_{i=1}^{\infty} \mu(U_i) = \mu\Big(\bigcup_{i=1}^{\infty} U_i\Big).$$

To prove (*), define the map $T : l^{\infty}(\mathbb{N}) \to l^{\infty}(I)$ by

$$T(a_1,a_2,\ldots) = \sum_{i=1}^{\infty} a_i \zeta(V_i).$$

Then T is a homomorphism and consequently $\Phi \circ T$ is a homomorphism. By using [v.R.-(6.29)], (*) follows at once. □

(1.34) REMARK. It is not difficult to show, if we believe in the existence of measurable cardinal numbers, that, when $\#I$ is measurable and $\#K$ is small, but K is not locally compact, we indeed can make homomorphisms $l^{\infty}(I) \to K$ not of the type mentioned in (1.33). If we assume that $\#I$ is measurable and K is such that $\#K \geq \#I$, then again we can prove the analogue of (1.33). □

CHAPTER II

STRUCTURE THEOREMS AND APPLICATIONS

In this chapter we prove that every torsional group G is in a natural way a topological direct sum of a closed p-primary subgroup G_1 and a closed p-free subgroup G_2 [see (2.1)]. We show that information about the Banach algebras $M(G_1)$ and $M(G_2)$ gives us information about the Banach algebra $M(G)$. [see (2.2) - (2.16)]. Finally, we show that there are two important types of compact p-primary groups [see (2.19)].

§1. A structure theorem for torsional groups.

(2.1) PROPOSITION. Let G be a torsional group. Put $G_1 = \{g \mid g \in G, \lim_{n\to\infty} p^n g = 0\}$ and $G_2 = \{g \mid g \in G, g \in [pg]\}$. Then G_1 and G_2 have the following properties:

(i) G_1 and G_2 are closed subgroups of G and G is the topological direct sum of G_1 and G_2;

(ii) G_1 is a p-primary group;

(iii) G_2 is a p-free group.

Proof. First we will prove that $G_1 \cap G_2 = \{0\}$ [see (i.a)]. Then we will prove that each compact subgroup $H \subset G$ is a topological direct sum of compact subgroups H_1 and H_2 [(i.b)] having the properties that $H_1 \subset H \cap G_1$ [(i.c)] and $H_2 \subset H \cap G_2$ [(i.d)]. Combining (i.a), (i.b), (i.c) and (i.d) we will prove that $H_1 = H \cap G_1$ and $H_2 = H \cap G_2$ [(i.e)]. Using these results, we will show that G is algebraically the direct sum of G_1 and G_2 [(i.f)]. Finally, we will show that G_1 and G_2 are closed subgroups [(i.g)] and that G is the topological direct sum of G_1 and G_2 [(i.h)].

(i.a) Let $g \in G_1 \cap G_2$. To prove the assertion it is sufficient to prove that $g \in U$ for each open subgroup $U \subset G$. Let $U \subset G$ be open. The fact that $g \in G_1$ tells us that there is an $n \in \mathbb{N}$ such that $\llbracket p^n g \rrbracket \subset U$. Further, since $g \in G_2$ we see now that $g \in \llbracket p^n g \rrbracket$, which completes the proof.

(i.b) Let $H \subset G$ be a compact subgroup. The topology on H is zerodimensional and so each element of $\hat{H}$ has finite order [H.R-(24.26)]. Put
$\Lambda_1 = \{\hat{h} \mid \hat{h} \in \hat{H}$, the order of $\hat{H}$ is a power of $p\}$ and
$\Lambda_2 = \{\hat{h} \mid \hat{h} \in \hat{H}$, the order of $\hat{H}$ is not divisible by $p\}$.
Then from [H.R-(A.3)] we infer that $\hat{H}$ is the direct sum of Λ_1 and Λ_2.
Let $H_1 = \{x \mid x \in H,\ \hat{h}(x) = 1$ all $\hat{h} \in \Lambda_2\}$ and
$H_2 = \{x \mid x \in H,\ \hat{h}(x) = 1$ all $\hat{h} \in \Lambda_1\}$. Now use [H.R-(23.18)] and we see that H is the topological direct sum of the compact subgroups H_1 and H_2.

(i.c) Let $g \in H_1$. To prove that $g \in H \cap G_1$ it suffices to prove that for each open subgroup $U \subset G$ there is an $n \in \mathbb{N}$ such that $p^n g \in U$. Let $U \subset G$ be open. From the fact that the dual group of $H_1/H_1 \cap U$ is isomorphic to a subgroup of $\hat{H}_1$ and the fact that $H_1/H_1 \cap U$ is finite it follows that $H_1/H_1 \cap U$ is a finite p-primary group. Consequently, there is an $n \in \mathbb{N}$ such that $p^n g \in H_1 \cap U \subset H \cap U$ and we are done.

(i.d) Let $g \in H_2$ and suppose $g \notin \llbracket pg \rrbracket$. This implies that we can embed C_p as a group in $\llbracket g \rrbracket / \llbracket pg \rrbracket$. Then some duality arguments show that there is a quotient of the group $\hat{H}_2$ that is isomorphic to C_p. There are no elements of order p in $\hat{H}_2$ and accordingly there are no elements of order p in such a quotient. Which means that we have a contradiction.

(i.e) Let $g \in H \cap G_1$. Then there are $g_1 \in H_1$ and $g_2 \in H_2$ such that $g = g_1 + g_2$ [(i.b)]. From (i.c) and (i.d) we deduce $g_2 = g - g_1 \in G_1 \cap G_2$. Combining this result with (i.a) we see $g_2 = 0$. Therefore $g \in H_1$ and we are done. One proves that $H_2 = H \cap G_2$ in completely analogous manner.

(i.f) $G_1 \cap G_2 = \{0\}$. So the only thing to prove is that $G = G_1 + G_2$. Let $g \in G$. Choose a compact subgroup $H \subset G$ for which $g \in H$. Now the proof follows immediately from (i.b) and (i.e).

(i.g) Let $g, h \in G_1$ [respectively G_2]. Choose a compact subgroup H such that $g, h \in H$. Use (i.b) and (i.e) and we get $g - h \in G_1$ [respectively G_2]. Consequently G_1 and G_2 are subgroups of G.

Next we show that G_1 is closed. Let $g \in \text{clo } G_1$. It suffices to prove that for each open subgroup $U \subset G$ there is an $n \in \mathbb{N}$ such that $p^n g \in U$. Therefore, choose an open subgroup $U \subset G$. There is a $g' \in G_1$ with $g - g' \in U$. Further, there is an $n \in \mathbb{N}$ such that $p^n g' \in U$. Consequently,

$$p^n g = p^n(g-g') + p^n g' \in U$$

and we are done.

The last thing to prove is that G_2 is closed. Assume that $\text{clo}(G_2) \cap G_1 = \{0\}$. Then take $g \in \text{clo } G_2$ and let $g_1 \in G_1$ and $g_2 \in G_2$ be such that $g = g_1 + g_2$. It follows that $g_1 = g - g_2 \in \text{clo } G_2$. Therefore, $g_1 = 0$ and we see that $g \in G_2$. So, what is left is the proof of the assumption that $\text{clo } G_2 \cap G_1 = \{0\}$. Take $g \in \text{clo } G_2 \cap G_1$. To prove that $g = 0$, it is enough to prove that $g \in U$ for each open subgroup $U \subset G$. Let $U \subset G$ be open. There is a $g' \in G_2$ such that $g - g' \in U$. Let $n \in \mathbb{N}$ be such that $p^n g \in U$. Then we infer $p^n g' = p^n(g'-g) + p^n g \in U$. $g' \in [\![p^n g']\!]$ and we get $g' \in U$. Therefore $g \in U$ and we are done.

(i.h) The continuity of the map $G_1 \times G_2 \to G$ is trivial. To prove the converse, let $U \subset G$ be an open subgroup and let $g \in U$. Choose a compact subgroup H such that $g \in H$. Then the group $H \cap U$ is compact and $g \in H \cap U$. Now use (i.b) and (i.e) to infer that $g_1 \in H \cap U$ and that $g_2 \in H \cap U$ [where $g_1 \in G_1$ and $g_2 \in G_2$ are such that $g = g_1 + g_2$]. This proves the continuity of the map $G \to G_1 \times G_2$ at 0, which is enough to prove the continuity of the map $G \to G_1 \times G_2$.

ii. Trivial

iii. Let $H \subset G_2$ be a compact subgroup and let $H_0 \subset H$ be a relatively open subgroup. We can identify the dual groups of H/H_0 with a finite subgroup of $\hat{H}$. Now use the construction in (i.b) to see that there are no elements of order p in $\hat{H}$. Therefore H/H_0 has no elements of order p, which means exactly that $[H : H_0]$ is not divisible by p and we are done. $\square$

§2. Structure theorems for the measure algebra M(G).

In this section we give some structure theorems for the measure algebra $M(G)$ [(2.3), (2.4), (2.8), (2.9).] (2.4) and (2.9) express $M(G)$ in terms of the measure algebras $M(G_1)$ and $M(G_2)$, where G_1 and G_2 are as in proposition (2.1).

Let us start with a definition.

(2.2) DEFINITION. Let G be a p-free compact group. We define an ordering $\leq$ on the collection of all idempotents of C(G) as follows: $e \leq e'$ if $e * e' = e$. Then a non-zero idempotent is called a minimal idempotent if e is minimal with respect to this ordering.

Note that when e and e' are different minimal idempotents then $e * e'$ is an idempotent and $e * e' \leq e'$, $e * e' \leq e$. Hence $e * e' = 0$.

The proof of the following theorem can be found in [v.R-(8.18)].

(2.3) THEOREM (van Rooij). Let G be a p-free compact group. Let E be the set of all minimal central idempotents of C(G). For each $e \in E$ define $B_e \subset C(G)$ as $B_e \underset{D}{=} C(G) * e$. Then each B_e is a finite dimensional extension field of K, the spaces B_e form an orthogonal system of linear subspaces of C(G). There exist Banach algebra isomorphisms $A : C(G) \overset{\sim}{=} \bigoplus_{e \in E} B_e$

$$B : M(G) = \times_{e \in E} B_e$$

given by
$$(Af)_e = f * e \qquad [f \in C(G),\ e \in E]$$
$$(B\mu)_e = \mu * e \qquad [\mu \in M(G),\ e \in E] \qquad \square$$

(2.4) THEOREM. Let G be a compact group. Let G_1 and G_2 be as in proposition (2.1) and let E be the set of all minimal idempotents of $C(G_2)$. For each $e \in E$ define C_e by $C_e \underset{D}{=} M(G) * e$. Then:

(i) for each $e \in E$, C_e is a closed subalgebra of M(G);

(ii) there is a Banachalgebra isomorphism $T : M(G) \overset{\sim}{=} \times_{e \in E} C_e$ given by

$$(T\mu)_e = \mu * e \quad [\mu \in M(G),\ e \in E].$$

Proof (i) Using the fact that e is an idempotent in $C(G_2)$ and therefore is an idempotent in $M(G_2)$, hence, an idempotent in M(G), one proves directly that C_e is a closed subalgebra of M(G).

(ii) The only non trivial things to prove are the assertions that T is an isometry and that T is a surjection.

The spaces $C(G_1)$ and $C(G_2)$ both have orthonormal bases [v.R.-(5.22)]. Choose an orthonormal base $(f_i)_{i\in I}$ of $C(G_1)$. Further let $(g_j)_{j\in J}$ be an orthonormal base of $C(G_2)$ such that for each $j \in J$ there is exactly one $e' \in E$ with $g_j \in C(G_2) * e'$. [The fact that there is such a base follows from theorem (2.3) and [v.R.-(5.9)]]. Now use [Y.A.-(3.4.2)] to infer that $(f_i \otimes g_j)_{(i,j)\in I\times J}$ is an orthonormal base of $C(G)$. First let us prove that T is an isometry. Let $\varepsilon > 0$. Then there is an $e' \in E$ together with $f \otimes g \in (f_i \otimes g_i)_{(i,j)\in I\times J}$ such that $g \in C(G_2) * e'$ and $|\mu(f \otimes g)| + \varepsilon > ||\mu||$. It is easy to see that $(\mu * e')(f \otimes g) = \mu(f \otimes g)$ and we obtain : $||T\mu|| \geq ||\mu||$. To prove that $||T\mu|| \leq ||\mu||$ is almost trivial.

What is left is the proof that T is a surjection. For this purpose, let $(\mu_e)_{e\in E} \in \underset{e\in E}{\times} C_e$. Define the linear map $\mu \in M(G)$ as follows. Take $f \otimes g \in (f_i \otimes g_j)_{(i,j)\in I\times J}$. Let $e' \in E$ be the unique element of E for which $g \in C(G_2) * e'$. Define $\mu(f \otimes g)$ by $\mu(f \otimes g) \underset{D}{=} \mu_{e'}(f \otimes g)$. Now it is clear that

$$T\mu = (\mu_e)_{e\in E}$$

and the proof is complete. $\square$

(2.5) Let us consider the structure of the closed subalgebra C_e mentioned in (2.4). For each $e \in E$ again put $B_e = C(G_2) * e$ and define the map $T_e : C_e \to L(C(G_1), B_e')$ by

$$(((T_e)(\mu))(f))(g) = \mu(f \otimes g)$$

$$[\mu \in C_e,\ f \in C(G_1),\ g \in B_e].$$

Then for each $e \in E$, T_e is a Banach space isomorphism. For each $e \in E$, $L(C(G_1), B_e')$ can be made into a Banach module over $M(G_1)$ in the following way

$$(\mu \circledast S)(f) \underset{D}{=} S(\mu * f) \quad [S \in L(C(G_1), B_e'),\ f \in C(G_1),\ \mu \in M(G_1)].$$

Now let $e_1^*,\ldots,e_n^*$ be a base of B_e'. Define $A_1,\ldots,A_n \in L(C(G_1), B_e')$ as follows $((A_i)(f))(g) = f(0)e_i^*(g)$ $[f \in C(G_1),\ g \in B_e]$.
Choose $A \in L(C(G_1), B_e')$. Then for each $f \in C(G_1)$ there are coefficients $\lambda_1,\ldots,\lambda_n$ such that $Af = \lambda_1 e_1^* + \ldots + \lambda_n e_n^*$. Define μ_i by $\mu_i(f) = \lambda_i$ $[i=1,\ldots,n]$. It is clear that μ_i is an element of $M(G_1)$ $[i=1,\ldots,n]$ and that $A = \mu_1 \circ A_1 + \ldots + \mu_n \circ A_n$. Hence, for each $e \in E$, $L(C(G_1), B_e')$, and therefore C_e, is a finite dimensional Banach module over $M(G_1)$. □

(2.8) and (2.9) are generalizations for torsional groups of (2.3) and (2.4), respectively.

First let us set up some more terminology.

(2.6) DEFINITION. For a torsional group G, $O(G)$ is to be the set of all positive integers n for which G has an open subgroup such that one of the elements of G/U has order n.

Note that for a p-free group, $p \notin O(G)$. □

(2.7) DEFINITION. Let G be a torsional group. Then G is <u>compatible</u> with K if $p \notin O(G)$ and if for all $n \in O(G)$, K has n distinct n^{th} roots of unity. □

The proof of (2.8) can be found in [v.R.-(9.20)].

(2.8) THEOREM (van Rooij). Let G be a torsional group, compatible with K. Then there is a Banach algebra isomorphism $F : M(G) \cong BUC\ (G^* \to K)$ given by

$$(F\mu)(\alpha) = \int \alpha(x)d\mu(x) \qquad [\mu \in M(G),\ \alpha \in G^*]. \qquad □$$

(2.9) THEOREM. Let G be a torsionalgroup. Let G_1 and G_2 be as in proposition (2.1). Assume that G_2 is compatible with K. Put

$A = \{f \mid f \in BUC(G_2^\star \to M(G_1))$, for each $\varepsilon > 0$ one has a compact subgroup $H \subset G_1$ and a function $g \in BUC(G_2^\star \to M(G_1))$, such that $\operatorname{supp} g(\alpha) \subset H$ for each $\alpha \in G_2^\star$ and $||f(\alpha) - g(\alpha)|| < \varepsilon$ for each $\alpha \in G_2^\star\}$. Then we have the following:

(i) A is a closed subalgebra of $BUC(G_2^\star \to M(G_1))$;

(ii) $A = BUC(G_2^\star \to M(G_1))$ in the case where G_1 is compact;

(iii) there is a Banach algebra isomorphism $F : M(G) \overset{\sim}{=} A$ given by

$$((F\mu)(\alpha))(f) = \mu(\alpha \otimes f) \quad [\mu \in M(G),\ \alpha \in G_2^\star,\ f \in BC(G_1)].$$

Proof. (i) follows immediately from (iii). (ii) is trivial. So, the only thing to prove is (iii). For the proofs of the assertions that F is an homomorphism $M(G) \to A$ and that F is an isometry $M(G) \to A$ we just can copy the proof of (2.8) [v.r.-(9.20)]. In the case where G_1 is compact the surjectivity of F is again a copy of the proof of (2.8) [v.R.-(9.20)]. In the case where G_1 is non-compact, realize that for each compact $H \subset G_1$, $BUC(G_2^\star, M(H))$ is a closed subalgebra of A and the proof follows directly. □

§3. Applications.

(2.10) PROPOSITION. Assume K is non-locally compact. Let G be a p-free compact group. Put E the set of all minimal idempotents of $C(G)$, $B_e = C(G) * e$ and $E_1 = \{e \mid e \in E,\ \dim B_e = 1\}$. Let $\Phi : M(G) \to K$ be a non-trivial homomorphism. Then there is an $a \in E_1$ such that $\Phi(\mu) = \mu(a)$ $[\mu \in M(G)]$. Conversely, each $e \in E_1$ defines a homomorphism $M(G) \to K$ given by $\mu \to \mu(e)$.

Proof. First note that $e \in E$ is in E_1 if e is a continuous character $G \to K$. (2.3) tells us that the map $T : M(G) \to \times_{e\in E} B_e$ defined by $(T\mu)_e = \mu * e$ is a Banach algebra isomorphism. Hence, $\Phi \circ T^{-1} : \times_{e\in E} B_e \to K$ is a homomorphism.

Define $\Psi : l^{\infty}(E) \to \underset{e\in E}{\times} B_e$ by $\psi(\lambda_e)_{e\in E} = (\lambda_e . e)_{e\in E}$. Then it is obvious that Ψ is a homomorphism. Consequently, $\Phi \circ T^{-1} \circ \Psi$ defines a homomorphism $l^{\infty}(E) \to K$. Now use [(1.33)] to deduce that there is an $a \in E$ such that $(\Phi \circ T^{-1} \circ \Psi)((\lambda_e)_{e\in E}) = \lambda_a$ [each $(\lambda_e)_{e\in E} \in l^{\infty}(E)$]. Let $\mu \in M(G)$. Then

$\Phi(\mu) = (\Phi \circ T^{-1})((\mu * e)_{e\in E}) = (\Phi \circ T^{-1})((\mu * e)_{e\in E}.\Psi(\zeta(E))) =$

$(\Phi \circ T^{-1})(\mu * e)_{e\in E}.\Psi(\zeta(E\backslash\{a\})) + \zeta(\{a\})) =$

$(\Phi \circ T^{-1})(\Psi(\zeta(E\backslash\{a\})))\cdot(\Phi \circ T^{-1})((\mu * e)_{e\in E})+(\Phi \circ T^{-1})(\Psi(\zeta(\{a\}))\cdot(\Phi \circ T^{-1})((\mu * e)_{e\in E}).$

Now use that fact that $(\Phi \circ T^{-1})(\Psi(\zeta(E\backslash\{a\}))) = 0$ and we infer $\Phi(\mu) = \Phi(\mu * a)$. Thus, Φ determines a homomorphism of $B_a \to K$. B_a is a finite dimensional extension field of K and Φ is non-trivial. Therefore, B_a must be one dimensional and consequently a is a continuous character. Hence, $\Phi(\mu) = \Phi(\mu * a) = \Phi(\mu(a)\cdot a) = \mu(a)\Phi(a)$. a is an idempotent and Φ is non-trivial, so $\Phi(a) = 1$. We conclude that $\Phi(\mu) = \mu(a)$ and the proof is done.

The proof of the assumption that each $e \in E_1$ determines a homomorphism is left to the reader. □

(2.11) PROPOSITION. Assume that K is locally compact. Let G be a p-free compact group. The set of all minimal idempotents of $C(G)$ we shall denote by E. For each $e \in E$, let $B_e = C(G) * e$. Put $E_1 = \{e \mid e \in E, \dim B_e = 1\}$. Let $\Phi : M(G) \to K$ be a homomorphism. Then there is an ultrafilter M on E_1 such that $\Phi(\mu) = \lim_{e\to M} \mu(e)$ $[\underset{D}{=} \bigcap_{A\in M} \mathrm{clo}\{\mu(e) \mid e \in A\}]$. Conversely, each ultrafilter N on E_1 determines an homomorphism in this way.

<u>Proof</u>. Let $E_2 = \{e \mid e \in E, \dim B_e > 1\}$. Let

$A_1 = \{(\mu * e)_{e\in E} \mid (\mu * e)_{e\in E} \in \underset{e\in E}{\times} B_e, \ \mu * e = 0$ for each $e \in E_2\}$ and

$A_2 = \{(\mu * e)_{e\in E} \mid (\mu * e)_{e\in E} \in \underset{e\in E}{\times} B_e, \ \mu * e = 0$ for each $e \in E_1\}$.

Let $T : M(G) \to \underset{e\in E}{\times} B_e$ be the Banach algebra isomorphism as defined in (2.3). Define $\Psi_1 : l^\infty(E_1) \to \underset{e\in E}{\times} B_e$ by

$$((\Psi_1)((\lambda_e)_{e\in E_1}))_d = 0 \quad \text{when } d \in E_2$$
$$= \lambda_d \cdot d \text{ when } d \in E_1.$$

It is clear that Ψ_1 and Ψ_2 are Banach algebra isomorphisms of $l^\infty(E_1)$ into A_1 respectively of $\underset{e\in E_2}{\times} B_e$ into A_2. Further it is obvious that

$$A_1 + A_2 = \underset{e\in E}{\times} B_e$$

and

$$A_1 \cap A_2 = \{0\}.$$

Now assume that the homomorphism $\Phi \circ T^{-1} \circ \Psi_2 : \underset{e\in E_2}{\times} B_e \to K$ is the zero-homomorphism.

Then

$$\Phi(\mu) = (\Phi \circ T^{-1})((\mu * e)_{e\in E}) =$$
$$(\Phi \circ T^{-1})((\mu_1 * e)_{e\in E} + (\mu_2 * e)_{e\in E})$$

[where

$(\mu_1 * e)_{e\in E} \in A_1$, $(\mu_2 * e)_{e\in E} \in A_2$ and $(\mu_1 * e)_{e\in E} + (\mu_2 * e)_{e\in E} = (\mu * e)_{e\in E}$]

$$= (\Phi \circ T^{-1} \circ \Psi_1)((\mu_1(e)_{e\in E_1}) + (\Phi \circ T^{-1} \circ \Psi_2)((\mu_2 * e)_{e\in E_2})$$

$= (\Phi \circ T^{-1} \circ \Psi_1)((\mu_1(e))_{e\in E_1})$. The homomorphisms $l^\infty(E_1) \to K$ correspond to ultrafilters on E_1 [(1.31)] and we see that there is an ultrafilter M on E_1 with

$$\Phi(\mu) = \lim_{e\to M} \mu_1(e).$$

What is left, is the proof of the claim that $\Phi \circ T^{-1} \circ \Psi_2 = 0$. Suppose $\Phi \circ T^{-1} \circ \Psi_2 \neq 0$. For each B_e $(e \in E_2)$ we can choose an element $\mu_e \in B_e$ such that $||\mu_e - \lambda.e|| \geq 1$ [all $\lambda \in K$][use [v.R.-(5.9)]]. The element

$(\mu_e)_{e\in E_2}$ is invertible in $\underset{e\in E_2}{\times} B_e$ [Each B_e is a finite dimensional extension field of K!] and we get that $(\Phi \circ T^{-1} \circ \Psi_2)((\mu_e)_{e\in E_2}) = \lambda \neq 0$. Using the same argument as above one sees that the element $(\mu_e - \lambda.e)_{e\in E_2}$ is invertible. Consequently, $0 = \Phi((\mu_e - \lambda.e)_{e\in E_2}) \neq 0$ and we have a contradiction. It is clear that each ultrafilter on E_1 defines a homomorphism $M(G) \to K$. □

(2.12) THEOREM. Assume that K is not locally compact. Let G be a compact group. Let G_1 and G_2 be as in proposition (2.1). Let E be the set of all minimal idempotents of $C(G_2)$. Put $E_1 = \{e \mid e \in E_1, \dim(C(G_2) * e) = 1\}$. For each $e \in E_1$ and $\mu \in M(G)$ define $\mu_e \in M(G_1)$ by

$$(\mu_e)(g) \underset{D}{=} \mu(g \otimes e) \qquad [g \in C(G_1)].$$

Let $\Phi : M(G) \to K$ be a homomorphism. Then there is an $a \in E_1$ such that $\Phi(\mu) = \Phi(\mu_a)$. Conversely, for each $e \in E_1$ and each homomorphism $\Psi : M(G_1) \to K$ the map $\mu \to \Psi(\mu_e)$ $[\mu \in M(G)]$ determines a homomorphism $M(G) \to K$.

<u>Proof</u>. Each $e \in E_1$ is a continuous character and therefore $\mu_e = \mu * e$. From (2.10) we deduce that there is an $a \in E_1$ such that

$$\Phi(\mu) = \mu(a) \qquad [\text{each } \mu \in M(G_2)].$$

Consequently,

$$\Phi(\mu) = \Phi(\bar{0}) \ . \ \Phi(\mu) = \Phi(a)\Phi(\mu) = \Phi(\mu * a) = \Phi(\mu_a).$$

The proof of the converse is straight forward. □

(2.13) COROLLARY. Assume that K is not locally compact. Let G be a torsional group and let G_1 and G_2 be as in proposition (2.1). For each c-continuous character $\alpha : G_2 \to K$ and each $\mu \in M(G)$ define $\mu_\alpha \in M(G_1)$

by $\mu_\alpha(f) = \mu(f \otimes \alpha)$ $[f \in BC(G_1)]$. Let Φ be an homomorphism $M(G) \to K$. Then there is a c-continuous character $\beta : G_2 \to K$ such that $\Phi(\mu) = \Phi(\mu_\beta)$ [all $\mu \in M(G)$]. Conversely, for each c-continuous character $\alpha : G_2 \to K$ and each homomorphism $\Psi : M(G_1) \to K$ the map $\mu \to \Psi(\mu_\alpha)$ is an homomorphism $M(G) \to K$.

Proof. Define α by $\alpha(x) = \Phi(\bar{x})$ $(x \in G_2)$. It is obvious that α is a character. Choose a compact subgroup H in G. Put H_1 and H_2 as in proposition (2.1). It follows from (2.12) that the map $x \to \Phi(\bar{x})$ $[x \in H]$ is continuous and we infer α is c-continuous. Moreover, it follows from (2.12) that $\Phi(\nu) = \Phi(\nu_\alpha)$ for each $\nu \in M(G)$ with supp ν compact. Now let $\mu \in M(G)$ and $\varepsilon > 0$. Choose $\mu' \in M(G)$ with supp μ' compact and $\|\mu - \mu'\| < \varepsilon$. Then

$$|\Phi(\mu) - \Phi(\mu_\alpha)| = |\Phi(\mu) - \Phi(\mu') + \Phi(\mu') - \Phi(\mu_\alpha)| \leq$$

$$\max(\|\Phi\| \, \|\mu - \mu'\|, \|\Phi\| \, \|\mu'_\alpha - \mu_\alpha\|) \leq \varepsilon.$$

Consequently, $\Phi(\mu) = \Phi(\mu_\alpha)$.

The proof of the converse is again straight forward. □

(2.14) THEOREM. Assume that K is locally compact. Let G be a compact group. G_1 and G_2 are as in Proposition (2.1). Put E and E_1 as in (2.11). For each $e \in E_1$, $\mu \in M(G)$ define $\mu_e \in M(G_1)$ as in (2.12). Let $\Phi : M(G) \to K$ be a homomorphism such that Ker $\Phi \cap M(G_1)$ is finitely generated as an ideal in $M(G_1)$. Then there is an ultrafilter M on E_1 such that

$$\Phi(\mu) = \lim_{e \to M} \Phi(\mu_e) \left[\overset{=}{D} \bigcap_{A \in M} \text{clo}\{\Phi(\mu_e) \mid e \in A\}\right] \text{ [each } \mu \in M(G)].$$

Proof. For each $e \in E$ define C_e as in (2.4). Let $T : M(G) \to \times_{e \in E} C_e$ be the Banach algebra isomorphism as defined in (2.4). For each $\mu \in M(G)$ define $((P\mu) * e)_{e \in E} \in \times_{e \in E} C_e$ by

$$(P\mu) * e = 0 \quad \text{when } e \notin E_1$$
$$= \mu * e \text{ when } e \in E_1.$$

From the proof of (2.11) we infer that $\Phi(\mu) = (\Phi \circ T^{-1})(((P\mu) * e)_{e\in E})$ $[\mu \in M(G)]$. Let $\nu_1,\ldots,\nu_n$ be generators of Ker $\Phi \cap M(G_1)$. Then for each $\mu \in M(G)$, there are $\rho_1,\ldots,\rho_n \in M(G)$ such that

$$((P\mu)*e)_{e\in E} = ((P\rho_1)*(P\nu_1)*e+\ldots+(P\rho_n)*(P\nu_n)*e+\Phi(\mu*e)(P\bar{O})*e)_{e\in E}.$$

Consequently,

$$\Phi(\mu) = \Phi(\nu_1)\Phi(\rho_1)+\ldots+\Phi(\nu_n)\Phi(\rho_n)+(\Phi\circ T^{-1})\ ((\Phi(\mu*e).(P\bar{O})*e)_{e\in E}) =$$
$$(\Phi \circ T^{-1})((\Phi(\mu * e).e)_{e\in E}).$$

Now use (2.11) to see that there is an ultrafilter M on E_1 such that

$$\Phi(\mu) = \lim_{e\to M} \Phi(\mu_e) \quad \text{for each } \mu \in M(G). \quad \square$$

(2.15) PROBLEM. What can be said about the homomorphisms $M(G) \to K$, in the case where K is locally compact, in terms of homomorphisms $M(G_1)\to K$ and $M(G_2) \to K$?

Another theorem that we can give in this context is the following.

(2.16) THEOREM. Assume that K is not spherically complete. Let G be a torsional group. Let $\Phi : M(G) \to K$ be a homomorphism. Then there is a c-continuous character $\alpha : G \to K$ such that $\Phi(\mu) = \mu(\alpha)$ [each $\mu \in M(G)$]. Conversely, each c-continuous character $G \to K$ determines a homomorphism $M(G) \to K$ in this way.

<u>Proof</u>. The dual space of M(G) is isomorphic to $BC_c(G)$ [v.R.-(7.25)] and it follows that the function α on G, defined by $\alpha(x) \underset{D}{=} \Phi(\bar{x})$, is a c-continuous character. The rest of the proof follows directly. The proof of the converse is again straight forward. $\square$

§4. Structure theorems for a compact p-primary group.

In this section we show that each zerodimensional p-primary compact group is a semidirect product of a group of type Z_p and a group of type C.

Groups of type Z_p are studied in chapter III, groups of type C in chapter IV, chapter V and chapter VI.

(2.17) DEFINITION. A torsional group G is called of type Z_p if for each compact subgroup $H \subset G$ there is an index set I such that $H \cong Z_p^I$. □

(2.18) DEFINITION. A compact group is called of type C if there is an index set J together with a suitable family $(n(j))_{j \in J}$ of positive integers such that $G \cong \prod_{j \in J} C_{p^{n(j)}}$. □

(2.19) THEOREM. Let G be a compact p-primary group. Then there is a closed subgroup H of G such that

(i) H is of type Z_p;

(ii) G/H is of type C.

Proof. From the fact that G is a zerodimensional p-primary group we deduce that $\hat{G}$ is a p-primary torsion group [see (2.1), parts (i.b) and (i.c)]. From [H-R-(A.24)] we infer that there is a subgroup $\Lambda \subset \hat{G}$ such that $\Lambda \cong \bigoplus_{j \in J} C_{p^{n(j)}}$ for some index set J and suitable family $(n(j))_{j \in J} \subset \mathbb{N}$ and such that $\hat{G}/\Lambda$ is divisible. $\hat{G}/\Lambda$ is also a p-primary group. Use this fact and [H-R-(A.14)] to deduce that $\hat{G}/\Lambda \cong \bigoplus_{i \in I} Z(p^\infty)$ for some index set I.
Let $H = \{x \mid x \in G, \gamma(x) = 1 \text{ all } \gamma \in \Lambda\}$.
Then some duality arguments show that H is of type Z_p and G/H is of type C. [The dual of $Z(p^\infty)$ is isomorphic to Z_p!] □

(2.20) REMARK. Assume that the set $\{n(j) \mid j \in J\}$ mentioned in the proof of (2.19) is bounded. Then $G \cong G/H \times H$.

Proof. H-R-(A.24) tells us that we may assume that the group Λ mentioned in the proof of (2.19) is pure. Then from [F-(27.5)] we deduce that $\hat{G} \cong \hat{G}/\Lambda \times \Lambda$.

By using a duality argument the statement follows. □

(2.21) THEOREM. Let G be a compact p-primary group.

Let $H_1 = \mathrm{clo}\{x \mid x \in G, x \text{ has finite order}\}$. Then there is a closed subgroup H_2 of G such that H_2 is of type Z_p and such that G is topologically the direct sum of H_1 and H_2.

Proof. G/H_1 is a p-primary group without elements of finite order. Therefore G/H_1 is isomorphic to Z_p^I for some index set I [see (3.1)]. Hence, $\Lambda_1 = \{\gamma \mid \gamma \in \hat{G}, \gamma(x) = 1 \text{ all } x \in H_1\}$ is a divisible subgroup of $\hat{G}$ $[\Lambda_1 \underset{D}{\cong} \bigoplus_I Z(p^\infty)!\,]$. From [F-(21.2)] it follows that there is a subgroup Λ_2 of $\hat{G}$ such that $\Lambda_1 \oplus \Lambda_2 = \hat{G}$.

Now by using some duality arguments it follows easily that there is a subgroup H_2 of G with $H_2 \cong G/H_1 \cong Z_p^I$ such that $G = H_1 \oplus H_2$. □

(2.22) REMARK. Suppose that the group H_1 in (2.21) is metrizable. Then H_1 is of type C.

Proof. It suffices to prove that $\hat{H}_1 \cong \bigoplus_{j \in \mathbb{N}} C_{p^{n(j)}}$ for a suitable family $(n(j))_{j \in \mathbb{N}}$ of integers.

Now H_1 is metrizable and therefore $\hat{H}_1$ is countable. Further, the fact that the elements of finite order are dense in H_1, tells us that each element of $\hat{H}_1$ is of finite height [i.e.: $g \in \hat{H}_1$, $g \neq 0$, then $\sup\{n \mid$ the equation

$p^n x = g$ is solvable in $H_1\}$ is finite].

Then from [F-(17.3)] it follows that $\hat{H}_1$ is a direct sum of cyclic groups.

$\square$

(2.23) REMARK. In general it is not true that a compact p-primary group is a (topological) direct sum of a group of type C and a group of type Z_p, as the following counterexample shows.

Counterexample. Let A be the (discrete) group consisting of the elements of $C_p \times C_{p^2} \times C_{p^3} \ldots$ that have finite order. Then A is a p-primary group without elements of infinite height [see (F-(17.3)]. Therefore there is no subgroup of A isomorphic to $\bigoplus_I Z(p^\infty)$ for some index set I. (a)

A is not a direct sum of cyclic groups [also [F-(17.3)]]. (b)

By using (a) and (b) it follows that $\hat{A}$ is a compact p-primary group, which is not a direct sum of a group of type C and a group of type Z_p. $\square$

CHAPTER III

GROUPS OF TYPE Z_p

§1. Properties of the norm of M(G) with respect to multiplication.

(3.1) PROPOSITION. Let G be a p-primary torsional group. Then the following conditions are equivalent:

(i) G is of type Z_p;

(ii) there are no non-trivial elements of finite order in G.

Proof. (i) $\Rightarrow$ (ii). The fact that G is torsional tells us that for each $g \in G$ there is a compact subgroup $H \subset G$ such that $g \in H$. Now (i) $\Rightarrow$ (ii) follows at once.

(ii) $\Rightarrow$ (i). Let $H \subset G$ be a compact subgroup. There are no non-trivial elements of finite order in G, so by using [H.-R.-(24.23)] we conclude that $\hat{H}$ is a divisible group. From the fact that H is a zerodimensional, compact p-primary group we infer that $\hat{H}$ is a p-primary torsion group.

Combining both results above and [H.-R.-(A.14)] we get that $\hat{H} \cong \bigoplus_I Z(p^\infty)$ for some index set I. Consequently, $H \cong (Z_p)^I$ and the proof is done. □

For a torsional group G, to say "G is a p-primary group" is the same as to say "For each compact subgroup $H \subset G$ and each relatively open subgroup $H_0 \subset H$, $[H:H_0] = p^n$ for some suitable $n \in \mathbb{N}$."[Use proposition (2.1), especially the construction in (i.b)]. So, we can reformulate our proposition in the following way.

(3.1)' PROPOSITION. Let G be a torsional group. Then the following conditions are equivalent:

(i) G is of type Z_p;

(ii) (a) there are no non-trivial elements of finite order in G,

(b) for each compact subgroup H of G and each relatively open subgroup H_0 of H the index $[H:H_0]$ is a power of p. $\square$

(3.2) DEFINITION. For each $n \in \mathbb{N}$ we consider the functions $g_n : Z_p \to Z_p$ given by $g_n(x) = \frac{x(x-1)\dots(x-n+1)}{n!}$ $[x \in Z_p]$. The function $g_0 : Z_p \to Z_p$ is given by $g_0(x) = 1 [x \in Z_p]$. The natural map from Z_p onto $\mathbb{F}_p$ is denoted by $x \to x^{\sim}$ $[x \in Z_p]$. We define a function $\binom{\cdot}{n} : Z_p \to K$.

$$\binom{x}{n} \underset{D}{=} g_n(x) \quad [x \in Z_p,\ n \in N \cup \{0\}] \text{ when } \mathbb{Q}_p \subset K.$$

$$\binom{x}{n} \underset{D}{=} g_n(x)^{\sim} \quad [x \in Z_p,\ n \in \mathbb{N} \cup \{0\}] \text{ when } \mathbb{F}_p \subset K.$$

Now we have the following well-known proposition that we give without proof. For remarks concerning the proof see for instance [v.R.- (5.29)] or [Y.A.- (3.2.2.2)] .

(3.3) PROPOSITION. Let $m \in \mathbb{N}$. Then the functions $f_{n(1),\dots,n(m)} : Z_p^m \to K$ [where $(n(1),\dots,n(m)) \in (\mathbb{N} \cup \{0\})^m$]

defined by

$$f_{n(1),\dots,n(m)}(x) = \prod_{i=1}^{m} \binom{x(i)}{n(i)} \qquad [x \in Z_p^m]$$

form an orthonormal base in $C(Z_p^m)$. $\square$

Let I be an index set. For a finite set $J \subset I \times \mathbb{N}$ define f_J by

$f_J(x) = 1 \quad [x \in Z_p^I]$ when $J = \emptyset$

$f_J(x) = \prod_{(i,n)\in J} \binom{x(i)}{n}$ when $J \neq \emptyset$

[Especially $f_{\{(i,n)\}} = \binom{\cdot}{n}$ when $\#I = 1$.]

Then

(3.4) PROPOSITION. The set $B \underset{D}{=} \{f_J \mid J \subset I \times \mathbb{N},\ J \text{ finite}\}$ is an orthonormal base of $C(Z_p^I)$.

Proof. The linear span of characteristic functions of clopen sets is dense in $C(Z_p^I)$. Hence, for the proof of the completeness of B it suffices to prove that each characteristic function of a clopen set is in the closure of the linear span of B. Let $O \subset Z_p^I$ be clopen. There is a finite sequence of elements $i(1),\dots,i(k)$ of I together with clopen sets $O_1,\dots,O_k$ of Z_p such that $O = \{x \mid x \in Z_p^I, x(i(j)) \in O_j,\ j = 1,\dots,k\}$. Let $\pi_1 : I \times \mathbb{N} \to I$ be the projection onto the first coordinate. Now it follows from (3.3) that $\zeta(O)$ is in the closure of the linear span of the functions $f_J \in B$ such that $\pi_1(J) \subset \{i(1),\dots,i(k)\}$. For the proof of the assertion that the set B is orthonormal take $f_{J_1},\dots,f_{J_m} \in B$ and $\alpha_1,\dots,\alpha_m \in K$, choose $\{i(1),\dots,i(k)\} \subset I$ such that

$\pi_1(J_j) \subset \{i(1),\dots,i(k)\}$ for each $j \in \{1,\dots,m\}$.

Use theorem (3.3) to infer that there is $x \in Z_p^I$ with $x(i) = 0$ for each $i \notin \{i(1),\dots,i(k)\}$ and $\left|\sum_{j=1}^{m} \alpha_j f_{J_j}(x)\right| = \max_{1\le j\le m} |\alpha_j|$.

Now the assertion follows immediately. □

(3.5) THEOREM. Let G be a torsional group. Assume that there are closed subgroups G_1 and G_2 of G such that G is the topological direct sum of G_1 and G_2. Further assume that G_1 is of type Z_p. Then for each $\mu \in M(G_1)$ and each $\nu \in M(G)$ we have that $||\mu * \nu|| = ||\mu||\ ||\nu||$.

Proof. Case 1. G is compact. The map $S : M(G) \to L(C(G_1),M(G_2))$ given by

$$((S\nu)(f))(g) \underset{D}{=} \nu(f \otimes g) \qquad [\nu \in M(G),\ f \in C(G_1),\ g \in C(G_2)]$$

is a Banach space isomorphism. [See [P.I-(1.5)]]

Define "multiplication by an element of $M(G_1)$" in $L(C(G_1), M(G_2))$ as follows

$$(\mu \,\square\, A)(f) \underset{D}{=} A(\mu * f) \qquad [\mu \in M(G_1),\ A \in L(C(G_1),M(G_2)),\ f \in C(G_1)].$$

Then for $\mu \in M(G_1)$, $\nu \in M(G)$, $f \in C(G_1)$, $g \in C(G_2)$ we have that

$$((\mu \,\square\, S\nu)(f))(g) = ((S\nu)(\mu * f))(g) = \nu(\mu * f \otimes g) =$$

$$(\mu * \nu)(f \otimes g) = ((S(\mu * \nu))(f))(g).$$

Hence, what we have to prove is that $||\mu \,\square\, A|| = ||\mu||\ ||A||$ for each $\mu \in M(G_1)$ and $A \in L(C(G_1),M(G_2))$. From now on let $\mu \in M(G_1)$ and $A \in L(C(G_1),M(G_2))$ be fixed.

For each $f \in C(G_1)$ with $||f|| \leq 1$ we have

$$||(\mu \,\square\, A)(f)|| = ||A(\mu * f)|| \leq ||A||\ ||\mu * f|| \leq ||A||\ ||\mu||.$$

What is left is the proof of the assertion that $||\mu||\ ||A|| \leq ||\mu \,\square\, A||$.

G_1 is compact and of type Z_p, so we may assume that $G_1 = Z_p^I$ for some index set I.

<u>Case (1.a)</u>. I is finite. Then $G \cong Z_p^m$ for some $m \in \mathbb{N}$.

Let $B = \{f_{n(1),\ldots,n(m)} \mid (n(1),\ldots,n(m)) \in (\mathbb{N} \cup \{0\})^m\}$ be the orthonormal base of $C(Z_p^m)$ as defined in (3.3). Choose a lexicographic ordering in $(\mathbb{N} \cup \{0\})^m$.

In the case that de valuation is discrete, let $\underline{l}$ en $\underline{k}$ be minimal elements in this ordering such that

$$||\mu|| = |\mu(f_{\underline{l}})| \text{ and } ||A|| = ||A(f_{\underline{k}})||.$$

Then $||(\mu \,\square\, A)(f_{\underline{l}+\underline{k}})|| = ||A(\mu * f_{\underline{l}+\underline{k}})|| =$

$||A(\Sigma\{\mu(f_{\underline{s}})f_{\underline{t}} \mid \underline{s}, \underline{t} \in (\mathbb{N} \cup \{0\})^m, \underline{s} + \underline{t} = \underline{k} + \underline{l}\}|| = ||A||\ ||\mu||$

In the case that the valuation is non-discrete, choose $\varepsilon > 0$ and $r \in K$ with $1 - \varepsilon < |r| < 1$. Define μ_r and A_r by

$$\mu_r(f_{n(1)\ldots,n(m)}) = r^{n(i)+\ldots+n(m)}\mu(f_{n(1),\ldots,n(m)})[(n(1),\ldots,n(m))\in(\mathbb{N}\cup\{0\})^m]$$

and

$$A_r(f_{n(1),\ldots,n(m)}) = r^{n(1)+\ldots+n(m)}A(f_{n(1),\ldots,n(m)})[n(1),\ldots,n(m))\in(\mathbb{N}\cup\{0\})^m].$$

Then in an analogous manner as above we prove that

$||\mu_r \mathbin{\square} A_r|| = ||\mu_r||\ ||A_r||$. By letting ε tend to 0 it follows that $||\mu_r||$ tends to $||\mu||$, $||A_r||$ tends to $||A||$ and $||\mu_r \mathbin{\square} A_r||$ tends to $||\mu \mathbin{\square} A||$. Consequently, $||\mu \mathbin{\square} A|| = ||\mu||\ ||A||$.

<u>Case (1.b)</u>. I is infinite. Let B be the orthonormal base of $C(Z_p^I)$ as defined in (3.4). Choose $\varepsilon > 0$ and let f_{J_1} and f_{J_2} be elements of B such that

$$|\mu(f_{J_1})| + \varepsilon > ||\mu|| \quad\text{and}\quad ||A(f_{J_2})|| + \varepsilon > ||A||.$$

Let S be a finite subset of I with $\pi_1(J_1) \subset S$ and $\pi_1(J_2) \subset S$ [where $\pi_1 : I \times \mathbb{N} \to I$ is the projection on the first coordinate]. Define $\mu' \in M(G_1)$ by

$$\begin{aligned}\mu'(f_J) &= 0 \text{ when } \pi_1(J) \not\subset S \qquad [f_J \in B]\\ &= \mu(f_J) \text{ when } \pi_1(J) \subset S \qquad [f_J \in B].\end{aligned}$$

$A' \in L(C(G_1), M(G_2))$ is defined by

$$\begin{aligned}A'(f_J) &= 0 \text{ when } \pi_1(J) \not\subset S \qquad [f_J \in B]\\ &= A(f_J) \text{ when } \pi_1(J) \subset S \qquad [f_J \in B].\end{aligned}$$

When $\pi_1(J) \not\subset S$ then $(\mu' \mathbin{\square} A')(f_J) = 0$ $[f_J \in B]$ and when $\pi_1(J) \subset S$ then $(\mu' \mathbin{\square} A')(f_J) = (\mu \mathbin{\square} A)(f_J)$ $[f_J \in B]$

Hence, $||\mu' \mathbin{\square} A'|| \leq ||\mu \mathbin{\square} A||$. From case (1.a) it follows that $||\mu' \mathbin{\square} A'|| = ||\mu'||\ ||A'||$. Therefore, $||\mu'||\ ||A'|| \leq ||\mu \mathbin{\square} A||$.

Now by letting ε tend to 0 we deduce that $||\mu||\ ||A|| \leq ||\mu' \square A||$ and the proof is done.

Case 2. The general case. Let $\mu \in M(G_1)$ and $\nu \in M(G)$. Choose $\varepsilon > 0$. There is a compact subgroup H of G together with $\mu' \in M(G_1 \cap H)$ and $\nu' \in M(H)$ such that $||\mu - \mu'|| < \varepsilon$ and $||\nu - \nu'|| < \varepsilon$ [see (1.5)].
Then $||\mu * \nu - \mu' * \nu'|| \leq \max(\varepsilon||\nu||, \varepsilon.||\mu||)$. It follows by taking ε sufficiently small that

$$||\mu * \nu|| = ||\mu' * \nu'|| = ||\mu'||\ ||\nu'|| = ||\mu||\ ||\nu||. \qquad \square$$

(3.6) THEOREM. Let G be a torsional group. Then the following conditions are equivalent

(i) G is of type Z_p.

(ii) the norm in M(G) is multiplicative.

Proof. (i) ⇒ (ii) follows directly from theorem (3.5).
(ii) ⇒ (i). Let H be a compact subgroup of G and let H_1 and H_2 be as in proposition (2.1). Now suppose that H_2 is non-trivial. Then it follows from theorem (2.3) that there is an idempotent $e \in M(H_2)$ with $e \neq \bar{0}$ and $e \neq 0$. e is also an idempotent in M(G). The norm in M(G) is multiplicative. Consequently, $0 = ||(e - \bar{0}) * e|| = ||(e - \bar{0})||\ ||e|| \neq 0$, which is a contradiction. Therefore, $H = H_1$. Thus, H is a p-primary group, which is equivalent to saying that for each relatively open subgroup H_0 of H the index $[H : H_0]$ is a power of p [as we have already observed for example in (3.1)']. Suppose that H contains a non-trivial element g that has finite order. H is a p-primary group. Hence, we may assume that the order of g is p. Now consider the element $\mu = \sum_{n=0}^{p-1} \overline{ng}$ of M(G). Then $||\mu|| = 1$. But $||\mu^p|| < 1$ and we have a contradiction again. Now use proposition (3.1)' to see that $H \cong Z_p^I$ for some index set I and the proof is done. $\square$

(3.7) REMARK. For each index set I, $\mathbb{Q}_p^I$ is a torsional group of type Z_p. However, $\mathbb{Q}_p^I$ is not what one may call a standard example of such a group, in the sense that each group of Z_p is isomorphically embeddable in a $\mathbb{Q}_p^I$ for some index set I. For instance, let E be an infinite dimensional Banachspace over $\mathbb{Q}_p$. Then E is a torsional group of type Z_p [use (3.1)], but cannot be embedded in this way in a $\mathbb{Q}_p^I$.

Proof. Suppose that E is homeomorphically embeddable in $\mathbb{Q}_p^I$, where I is an index set. Denote this embedding by T and denote the projection of $\mathbb{Q}_p^I$ onto the i^{th} coordinate by P_i. Then for each $i \in I$, $P_i \circ T$ is a continuous group homomorphism from E onto $\mathbb{Q}_p$. It follows easily that $P_i \circ T$ is a continuous linear map from E onto $\mathbb{Q}_p$. Therefore, $(P_i \circ T)(B)$ [where B is the unit sphere in E] is a bounded set.

For each $\lambda \in (P_i \circ T)(B)$ and $\beta \in \mathbb{Q}_p$, $|\beta| \leq |\lambda|$, we have that $\beta \in (P_i \circ T)(B)$. Using this argument and the fact that the valuation on $\mathbb{Q}_p$ is discrete, it follows that $(P_i \circ T)(B)$ is closed in $\mathbb{Q}_p$. We deduce that $(P_i \circ T)(B)$ is compact and according to this fact, T(B) is compact in $\mathbb{Q}_p^I$. Consequently, B is compact, but this is a contradiction [E is an infinite dimensional vectorspace!]

§2. Homomorphisms $M(G) \to K$ in the case where G is of type Z_p.

(3.8) THEOREM. Let $m \in \mathbb{N}$ and let $\Phi : M(Z_p^m) \to K$ be a homomorphism. Then there is a unique continuous character $\alpha : Z_p^m \to K$ such that $\Phi(\mu) = \mu(\alpha)$ [$\mu \in M(Z_p^m)$]. Conversely, each continuous character $Z_p^m \to K$ determines an homomorphism in this way.

Proof. The map $S : M(Z_p^m) \to K||X_1,\dots,X_m||$ given by

$$S\mu = \sum_{n(1),\dots,n(m)} \mu\left(\prod_{i=1}^{m} \binom{x(i)}{n(i)}\right) \prod_{i=1}^{m} X_i^{n(i)}$$

is a Banach algebra isomorphism. [See [P- I]. Let $\beta(i) = (\Phi \circ S^{-1})(X_i)$ $[i = 1,\ldots,m]$. $\Phi \circ S^{-1}$ is a homomorphism $K||X_1,\ldots,X_m|| \to K$. Therefore,

$$(\Phi \circ S^{-1})(f) = \sum_{n(1),\ldots,n(m)} a(n(1),\ldots,n(m))\left(\prod_{i=1}^{m} (\beta(i))^{n(i)}\right)$$

for each $f \in K||X_1,\ldots,X_m||$, $f = \sum_{n(1),\ldots,n(m)} a(n(1),\ldots,n(m))\left(\prod_{i=1}^{m} X_i^{n(i)}\right)$ [see (1.28)].

Let $\alpha : Z_p^m \to K$ be defined by $\alpha(x(1),\ldots,x(m)) \underset{D}{=} \prod_{i=1}^{m} (1 + \beta(i))^{x(i)}$ [where the function $x \to \lambda^x$ is defined as in [Y.A.- p.91].

Then α is a continuous character $Z_p^m \to K$. Elementary calculations show that

$$\alpha(x) = \sum_{n(1),\ldots,n(m)} \left(\prod_{i=1}^{m} \beta(i)^{n(i)}\right)\left(\prod_{i=1}^{m} \binom{x(i)}{n(i)}\right) \quad [x \in Z_p^m].$$

We infer that for each $\mu \in M(Z_p^m)$

$$\Phi(\mu) = (\Phi \circ S^{-1}) \sum_{n(1),\ldots,n(m)} \mu\left(\prod_{i=1}^{m} \binom{x(i)}{n(i)}\right)\left(\prod_{i=1}^{m} X_i^{n(i)}\right) =$$

$$\sum_{n(1),\ldots,n(m)} \mu\left(\prod_{i=1}^{m}\binom{x(i)}{n(i)}\right)\left(\prod_{i=1}^{m} \beta(i)^{n(i)}\right) = \mu(\alpha).$$

The proof of the converse is straight forward. □

Denote $B_\infty \underset{D}{=} \{x \mid x \in l^\infty(\mathbb{N}),\ |x_n - 1| < 1$ for each n and $x_{n+1}^p = x_n$ for each $n\}$.

B_∞ is a group when we take as the group structure on B_∞ the restriction to B_∞ of multiplication in $l^\infty(\mathbb{N})$. Now let $\alpha : \mathbb{Q}_p \to K$ be a continuous character. Consider the sequence $(\alpha(1), \alpha(p^{-1}), \alpha(p^{-2}),\ldots)$. It is clear that $(\alpha(p^{-(n+1)}))^p = \alpha(p^{-n})$. The continuity of α tells us that $\lim_{n\to\infty} \alpha(p^{n-m}) = 1$ for each m. Hence, $|\alpha(p^{-m}) - 1| < 1$ for each m. Now it is trivial that the map $\alpha \to (\alpha(1),\alpha(p^{-1}),\alpha(p^{-2}),\ldots)$ is a group homomorphisms of $\mathbb{Q}_p^*$ into B_∞. Conversely, let $(\lambda_1,\lambda_2,\ldots) \in B_\infty$. Put

$\mathbb{Q}_p^0 = \{x \mid x \in \mathbb{Q}_p$, there are $m, n \in Z$ such that $x = m.p^{-n}\}$. The fact that $\lambda_{n+1}^p = \lambda_n$ for each $n \in \mathbb{N}$ tells us that we can define an $\alpha : \mathbb{Q}_p^0 \to K$ by the formula

$$\alpha(m.p^{-n}) = \lambda_n^m \qquad [m.p^{-n} \in \mathbb{Q}_p^0].$$

It is clear that $\alpha : \mathbb{Q}_p^0 \to K$ is a character. $\mathbb{Q}_p^0$ is dense in $\mathbb{Q}_p$. $|\lambda_m - 1| < 1$ for each $m \in \mathbb{N}$ and therefore $\lim_{n\to\infty} \alpha(p^{n-m}) = 1$ for each $m \in \mathbb{N}$. Hence, α can be extended by continuity to a continuous character on $\mathbb{Q}_p$. Now we infer the following proposition.

(3.9) PROPOSITION. The map $\alpha \to (\alpha(1), \alpha(p^{-1}), \ldots)$ is a group isomorphism of $\mathbb{Q}_p^*$ onto B_∞. $\square$

(3.10) REMARK. When we put on $\mathbb{Q}_p^*$ the topology of uniform convergence on compact sets and on B_∞ the restriction topology of the topology on $l^\infty(\mathbb{N})$, then the map given in (3.9) is an homeomorphism. $\square$

(3.11) THEOREM. Let $m,n \in \mathbb{N} \cup \{0\}$ and let $\Phi : M(\mathbb{Q}_p^m \times Z_p^n) \to K$ be a homomorphism. Then there is a continuous character $\alpha : \mathbb{Q}_p^m \times Z_p^n \to K$ such that $\Phi(\mu) = \mu(\alpha)$ (for each $\mu \in M(Z_p^n)]$.Conversely, each continuous character $\mathbb{Q}_p^m \times Z_p^n \to K$ determines an homomorphism in this way.

Proof. Denote $H_N = \{x \mid x \in \mathbb{Q}_p^m \times Z_p^n, |x_i| \le p^N, i=1,\ldots,m+n\}$. Define α by $\alpha(x) = \Phi(\bar{x})_D$ $[x \in \mathbb{Q}_p^m \times Z_p^n]$. Then α is a character. It follows from (3.8) that α is continuous on H_N for each N. Therefore, α is continuous on $\mathbb{Q}_p^m \times Z_p^n$. Let $\nu \in M(\mathbb{Q}_p^m \times Z_p^n)$ have compact support C. There is an $N \in \mathbb{N}$ such that $C \subset H_N$ and it follows from (3.8) that $\Phi(\nu) = \nu(\alpha)$. Now let $\mu \in M(\mathbb{Q}_p^m \times Z_p^n)$ be arbitrary. Take $\epsilon > 0$. Then there is a μ' having compact support such that $||\mu - \mu'|| \le \epsilon$. We deduce that

$$|\Phi(\mu) - \mu(\alpha)| = |\Phi(\mu) - \Phi(\mu') + \Phi(\mu') - \mu(\alpha)| = |\Phi(\mu) - \mu'(\alpha) + \Phi(\mu') - \mu(\alpha)|$$

$\leq \max\{||\Phi|| \; ||\mu - \mu'||, ||\alpha|| \; ||\mu - \mu'||\} \leq \varepsilon.$

Hence $\Phi(\mu) = \mu(\alpha)$. The proof of the converse is straight forward. $\square$

Let I be an index set and let $\lambda \in c_\infty(I)$, $\lambda = (\lambda(i))_{i\in I}$, be such that $||\lambda|| < 1$. Standard arguments show that we can define a continuous character α on Z_p^I by $\alpha(x) \underset{D}{=} \prod_i (1 + \lambda(i))^{x(i)}$ [where $(1+\lambda(i))^{x(i)}$ is defined as in [Y.A.-p.91]][$x \in Z_p^I$]. Conversely, let α be a continuous character on Z_p^I. Define $\lambda(i)$ by $\lambda(i) = \alpha(e_i) - 1$ [where for each $i \in I$, $e_i \in Z_p^I$ is defined by $e_i(j) = 0$ when $j \neq 0$ and $e_i(i) = 1$].
Then the continuity of α shows that $|\lambda(i)| < 1$ for each $i \in I$ and that $(\lambda(i))_{i\in I} \in c_\infty(I)$. We conclude that we have the following proposition.

(3.12) PROPOSITION. For each index set I there is a one to one correspondence between the continuous characters from Z_p^I to K and the $\lambda \in c_\infty(I)$ for which $||\lambda|| < 1$ via $\alpha(x) = \prod_i (1 + \lambda(i))^{x(i)}$ [$\lambda = (\lambda(i))_{i\in I}$, $x \in Z_p^I$]. $\square$

(3.13) REMARK. Let I be an infinite index set and let K be locally compact. Then there are homomorphisms $M(Z_p^I) \to K$, not of type $\mu \to \mu(\alpha)$ for some $\alpha : Z_p \to K$.

Proof. For notational reasons assume that $I = \mathbb{N}$. Choose $\lambda \in K$ such that $\lambda \neq 1$ and $|\lambda - 1| < 1$. For each $n \in \mathbb{N}$ define the continuous character α_n by $\alpha_n(x) = \lambda^{x(1)+\ldots+x(n)}$ [$x = (x(i))_{i\in\mathbb{N}} \in Z_p^{\mathbb{N}}$].
$\Psi : M(Z_p^{\mathbb{N}}) \to l_\infty(\mathbb{N})$ is the homomorphism given by

$$\Psi(\mu) = (\mu(\alpha_n))_{n\in\mathbb{N}} \quad [\mu \in M(Z_p^{\mathbb{N}})].$$

Let $\mathcal{M}$ be a free ultrafilter on $\mathbb{N}$ and let $\Phi : l^\infty(\mathbb{N}) \to K$ be the homomorphism corresponding to $\mathcal{M}$ [i.e. $\Phi((a_n)_{n\in\mathbb{N}})$ is the unique element of the set $\bigcap_{A \in M} \operatorname{clo}\{a_n | n \in A\}$].

For each $n \in \mathbb{N}$ let $e_n \in Z_p^{\mathbb{N}}$ be the following element

$e_n(m) = 0$ when $n \neq m$ and $e_n(n) = 1$.

Then $\bar{e}_n(\alpha_m) = 1$ when $m < n$ and $\bar{e}_n(\alpha_m) = \lambda$ when $m \geq n$, so, by using the fact that M is a free ultrafilter we see that $(\Phi \circ \Psi)(\bar{e}_n) = \lambda$ for each $n \in \mathbb{N}$. The sequence $(e_n)_{n \in \mathbb{N}}$ tends to 0. Therefore

$$(\Phi \circ \Psi)(\mu) = \mu(\alpha) \qquad [\text{all } \mu \in M(Z_p^{\mathbb{N}})]$$

for some continuous character $\alpha : Z_p^{\mathbb{N}} \to K$ would imply that

$$\lambda = \lim_{n\to\infty} (\Phi \circ \Psi)(\bar{e}_n) = \lim_{n\to\infty} (\bar{e}_n(\alpha)) = \lim_{n\to\infty} \alpha(e_n) = 1.$$

Which is a contradiction. $\square$

(3.14) PROBLEM. What can be said about the homomorphisms $M(Z_p^I) \to K$ in the case where K is spherically complete and not locally compact? [In the case that K is not spherically complete, see (2.16).]

(3.15) PROBLEM. What can in general be said about the homomorphisms $M(G) \to K$ in the case when G is a torsional group of type Z_p?

§3. Invertibility in groups of type Z_p.

(3.16) LEMMA. Let G be a compact p-primary group. Choose $\mu \in M(G)$ such that $||\mu|| = |\mu(G)| \neq 0$. Let $\alpha : G \to K$ be a character that has open kernel H. Then $||\mu|| = |\mu(\alpha)|$.

Proof. Choose $h_1,\ldots,h_n \in G$ such that $\bigcup_{i=1}^{n} (h_i+H) = G$ and such that $h_i - h_j \notin H$ when $i \neq j$.

G/H is a p-primary group. Hence, there is an $n \in \mathbb{N}$ with $(\alpha(h_i))^{p^n} = 1$ for each $i \in \{1,\ldots,n\}$. Consequently, $|1 - \alpha(h_i)|^{p^n} < 1$ for each

$i \in \{1,\ldots,n\}$. This implies especially that $|1 - \alpha(h_i)| < 1$ for each $i \in \{1,\ldots,n\}$. Then $|\mu(G) - \mu(\alpha)| = |\sum_{i=1}^{n} (1-\alpha(h_i))\mu(h_i + H)| \leq \max_{1\leq i\leq n} |1 - \alpha(h_i)|\ |\mu(h_i + H)| < ||\mu|| = |\mu(G)|$.

We infer that $|\mu(\alpha)| = |\mu(G)| = ||\mu||$. □

(3.17) LEMMA. Let G be a torsional group. L is a non-Archimedean valued complete extension field of K. Denote the Banach algebra over L of tight measures on G with values in L by $M_L(G)$. In a natural way $M(G)$ is a closed K-subalgebra of $M_L(G)$. Let $\mu \in M(G)$ be invertible in $M_L(G)$. Then μ is invertible in $M(G)$.

Proof. The invertible elements in $M(G)$ [$M_L(G)$] form an open set [v.R.-p.206]. Further the map $\rho \to \rho^{-1}$ is continuous and each element of $M(G)$ [$M_L(G)$] can be approximated by elements of $M(G)$ [$M_L(G)$] that have compact support. G is torsional. Consequently, we may assume that G is compact. For each clopen set O of G there is an open subgroup H of G such that O is a finite union of cosets of H. We infer that we are done when we prove the following: Let H be an open subgroup of G and let $h_1,\ldots,h_n \in G$ be such that G is the disjoint union of $h_1 + H,\ldots,h_n + H$, then $\mu^{-1}(h_i + H) \in K$ [where μ^{-1} denotes the inverse of μ].

Now $(\mu^{-1}(h_1+H),\ldots,\mu^{-1}(h_n+H))$ is the solution of the equations

$$(*)\qquad \begin{array}{l} \mu(h_1-h_1+H)\cdot x_1 + \mu(h_1-h_2+H)x_2+\ldots+\mu(h_1-h_n+H)x_n = 1 \\ \mu(h_2-h_1+H)\cdot x_1 + \mu(h_2-h_1+H)x_2+\ldots+\mu(h_2-h_n+H)x_n = 0 \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \mu(h_n-h_1+H)\cdot x_1 + \mu(h_n-h_2+H)x_2+\ldots+\mu(h_n-h_n+H)x_n = 0 \end{array}$$

μ is invertible in $M_L(G)$. Thus rank $((\mu(h_i - h_j + H))_{i,j=1,\ldots,n}) = n$. This means that $(*)$ has a solution in K^n. Further $(*)$ has a unique solution in L^n.

Hence, $(\mu^{-1}(h_1 + H),\ldots,\mu^{-1}(h_n + H)) \subset K^n$. $\square$

(3.18) **LEMMA.** Let G be a finite p-primary group. Let $\mu \in M(G)$ be such that $||\mu|| = |\mu(G)| \neq 0$. Then μ is invertible and $||\mu^{-1}|| = |\mu(G)|^{-1}$.

Proof. Let $G = \{g_1,\ldots,g_n\}$ [where $g_i \neq g_j$ when $i \neq j$ and $g_1 = 0$]. The proof in the case where characteristic K is finite, will be given in [(5.4)], so assume that $\mathbb{Q}_p \subset K$. Lemma (3.17) tells us that we may assume that K is algebraically closed. Another assumption that we of course may make, is that $|\mu(G)| = 1$.

Define $A : M(G) \to M(G)$ by $A\rho = \mu * \rho$ $[\rho \in M(G)]$.
An easy calculation shows that for each $\alpha \in G^*$, $\mu(\alpha)$ is eigenvalue of A with eigenvector $\sum_{i=1}^{n} \alpha(-g_i)\bar{g}_i$.
The fact that K is algebraically closed tells us that

$$\#G^* = \#G = \dim M(G).$$

Use (3.15) to deduce that

$$|\det A| = |\prod_{\alpha \in G^*} \mu(\alpha)| = 1.$$

It follows that A is a linear bijection and therefore that μ is invertible. To calculate

$$\mu^{-1}(h_1),\ldots,\mu^{-1}(h_n),$$

we have to solve the following set of equations

$$\begin{array}{cccc} \mu(h_1-h_1)x_1 + \mu(h_1-h_2)x_2 + \ldots + \mu(h_1-h_n)x_n & = 1 \\ \mu(h_2-h_1)x_1 + \mu(h_2-h_2)x_2 + \ldots + \mu(h_2-h_n)x_n & = 0 \\ \vdots & \vdots \\ \mu(h_n-h_1)x_1 + \mu(h_n-h_2)x_2 + \ldots + \mu(h_n-h_n)x_n & = 0 \end{array}$$

Use Cramer's rule to do this. Then use the relations $||\mu|| = 1$ and $|\det A| = 1$

to conclude that $|\mu^{-1}(h_i)| \leq 1$ for each $i \in \{1,\ldots,n\}$. It follows that $||\mu^{-1}|| \leq 1$. From the fact that $||\mu * \mu^{-1}|| = 1$ we infer that $||\mu^{-1}|| = 1$. $\square$

(3.19) PROPOSITION. Let G be a p-primary torsional group. Let $\mu \in M(G)$ be such that $||\mu|| = |\mu(G)| \neq 0$. Then μ is invertible and $||\mu^{-1}|| = |\mu(G)|^{-1}$.

Proof. The proof in the case where the characteristic of K is finite, will be given in [(5. 3)]. We assume that K is algebraically closed [see (3.16)]. Using the same arguments as in the proof of (3.16) we see that we may assume that G is compact.

Assume that the following is true:

Let H be an open subgroup of G and let $h_1,\ldots,h_n \in G$ be such that G is the disjoint union of $h_1 + H,\ldots,h_n+H$. Then we can define

$$\lambda(h_1+H),\ldots,\lambda(h_n+H) \in K$$

in such a way that

$$|\lambda(h_i+H)| \leq |\mu(G)|^{-1} \text{ for each } i \in \{1,\ldots,n\}$$

and such that

$$\begin{aligned} \Sigma_{j=1}^{n}\lambda(h_j+H)\mu(h_i-h_j+H) &= 1 \quad \text{when} \quad h_i + H = H \\ &= 0 \quad \text{when} \quad h_i + H \neq H \end{aligned}$$

[for each $i \in \{1,\ldots,n\}$].

Define $\mu^{-1}(h_i+H)$ by $\mu^{-1}(h_i+H) = \lambda(h_i+H)$. Now realize that each clopen set is a finite union of open cosets of G and the proposition follows at once.

Proof of the assumption. The natural map $\phi : G \to G/H$ induces an homomorphism $\Phi : M(G) \to M(G/H)$. Then $||\Phi(\mu)|| \leq ||\mu||$. But

$$|\Phi(\mu)(G/H)| = |\mu(G)| = ||\mu||.$$

We deduce that

$$||\Phi(\mu)|| = |\Phi(\mu)(G/H)| = |\mu(G)|.$$

Then according to (3.18), $\Phi(\mu)$ is invertible in $M(G/H)$ and for each $i \in \{1,\ldots,n\}$,

$$|(\Phi(\mu))^{-1}(\phi(h_i))| \leq |\mu(G)|^{-1}.$$

Define $\lambda(h_i+H)$ by

$$\lambda(h_i+H) = (\Phi(\mu))^{-1}(\phi(h_i)).$$

Then it is an easy exercise to show that $\lambda(h_1+H),\ldots,\lambda(h_n+H) \in K$ have the wanted properties.

(3.20) THEOREM. Let G be of type Z_p. Let $\mu \in M(G)$, $\mu \neq 0$. Then the following conditions are equivalent:

(i) μ is invertible;

(ii) $||\mu|| = |\mu(G)|$;

(iii) $|\mu(\alpha)| = ||\mu||$ for each c-continuous character $\alpha : G \to K$.

Proof. (i) $\Rightarrow$ (ii). The norm in $M(G)$ is multiplicative. [(3.6)]. Denote the inverse of μ by ν. Then $||\mu|| > |\mu(G)|$ implies that

$$||\nu|| = ||\mu||^{-1} < |\mu(G)|^{-1}.$$

But $|(\mu * \nu)(G)| = |\mu(G)|\ |\nu(G)| = 1$. Consequently,

$$|\nu(G)| = |\mu(G)|^{-1} > ||\nu||.$$

Which is a contradiction.

(ii) $\Rightarrow$ (i) follows from [(3.18)].

(iii) $\Rightarrow$ (ii) is trivial.

(i) $\Rightarrow$ (iii) Let α be a c-continuous character. Put $\rho = \alpha.\mu$.

Then ρ is invertible. It follows from the implication (i) $\Rightarrow$ (ii) that $||\rho|| = |\rho(G)| = |\hat{\mu}(\alpha)|$. Further,

$$||\mu|| \geq ||\rho|| = ||\alpha^{-1}|| \; ||\alpha\mu|| \geq ||\mu||.$$

Hence,

$$||\mu|| = ||\rho|| = |\mu(\alpha)|. \quad \square$$

In (3.21) we will show that the only torsional groups where we have equivalence of the properties (i) and (ii) mentioned in (3.18) are just the groups of type Z_p.

(3.21) THEOREM. Let G be a torsional group. Then the following conditions are equivalent:

(i) G is of type Z_p;

(ii) the fact that μ is invertible implies that $||\mu|| = |\mu(G)|$.

Proof. (i) $\Rightarrow$ (ii) is [(3.20)-i $\Rightarrow$ ii].

(ii) $\Rightarrow$ (i) We prove that G has the properties mentioned in [(3.1)'-(ii)].

(b) Let H be a compact subgroup of G. Let H_1 and H_2 be the subgroups of H as mentioned in proposition (2.1). Now suppose that there is a relatively open subgroup $H_0 \subset H$ such that $[H:H_0]$ is not a power of p. Then by using [(2.1)-(i.b)] we see that H_2 is non-trivial. Hence, there is a non-trivial minimal idempotent e, and therefore not invertible, in $C(H_2)$ such that $|e(G)| = ||e|| = 1$ [use (2.3)].

(a) Suppose there is a non-trivial element $g \in G$ such that g has finite order. It follows from (b) above that we may assume that g has order p.

Choose $\lambda \in K$ such that $0 < |1 - \lambda| < 1$ and such that λ is not a root of unity. Let $\mu = \bar{0} - \lambda\bar{g}$. Then an easy exercise shows that μ is invertible. But $1 = ||\mu|| > |1 - \lambda| = \mu(G)$, which is a contradiction. □

For convenience, let us collect some of the foregoing results in one statement.

(3.22) THEOREM. Let G be a torsional group. The following properties are equivalent:

(i) G is of type Z_p;

(ii) (a) there are no non-trivial elements of finite order in G,

(b) for each compact subgroup H of G and each relatively open subgroup H_0 of H the index $[H:H_0]$ is a power of p;

(iii) the norm in $M(G)$ is multiplicative;

(iv) for $\mu \in M(G)$, $\mu \neq 0$, the following conditions are equivalent.

(a) μ is invertible,

(b) $||\mu|| = \mu(G)$. □

Another theorem that we can give in this context is the following.

(3.23) THEOREM. Assume that K is algebraically closed. Let $G \cong Z_p^I$, where I is an index set. Let $\mu \in M(G)$, $\mu \neq 0$. Then the following assertions are equivalent.

(i) μ is invertible;

(ii) $||\mu|| = |\mu(G)|$;

(iii) $||\mu|| = |\mu(\alpha)|$ for each $\alpha \in G^*$;

(iv) $|\mu(\alpha)| = |\mu(G)|$ for each $\alpha \in G^*$

(v) $\mu(\alpha) \neq 0$ for each $\alpha \in G^*$.

<u>Proof</u>. The equivalence of (i), (ii) and (iii) has been proved in (3.19). (iii) ⇒ (iv) is trivial. (iv) ⇒ (v) follows when we realize that the algebra spanned by the continuous character is dense in $C(Z_p^I)$ [use the Kaplansky-Stone-Weierstrass theorem]. (i) ⇒ (v) is also trivial. So, when we can prove (v) ⇒ (ii) then we are done.

<u>Proof of</u> (v) ⇒ (ii).

<u>Case 1</u>. $\#I$ is finite. Then there is $m \in \mathbb{N}$ such that $G \cong Z_p^m$. Let $\{f_{n(1),\ldots,n(m)} \mid n(1),\ldots,n(m) \in (\mathbb{N} \cup \{0\})^m\}$ be the orthonormal base of $C(Z_p^m)$ as defined in (3.3) and let $S : M(Z_p^m) \to K||X_1,\ldots,X_m||$ be the Banach algebra isomorphism given by

$$S\mu = \sum_{n(1),\ldots,n(m)} \mu(f_{n(1),\ldots,n(m)})\left(\prod_{i=1}^{m} X_i^{n(i)}\right).$$

Suppose that $||\mu|| > |\mu(G)|$. Then there is $(k(1),\ldots,k(m)) \in (\mathbb{N} \cup \{0\})^m$ with

$$|\mu(f_{k(1),\ldots,k(m)})| > |\mu(f_{0,\ldots,0})|$$

From [(1.29)] it follows that there is $(\beta(1),\ldots,\beta(m)) \in (\Delta_1(0))^m$ such that

$$\Sigma\, \mu(f_{n(1),\ldots,n(m)})\left(\sum_{i=1}^{m} \beta(i)^{n(i)}\right) = 0.$$

Define $\alpha : Z_p^m \to K$ by $\alpha(x) = \prod_{i=1}^{m} (1 + \beta(i))^{x(i)}$.

Some easy calculations show that $\mu(\alpha) = \Sigma\, \mu(f_{n(1),\ldots,n(m)})\left(\prod_{i=1}^{m} \beta(i)^{n(i)}\right) = 0$ and we have a contradiction.

<u>Case 2</u>. $\#I$ is infinite. Let B be the orthonormal base of $C(Z_p^I)$ as chosen in (3.4). $||\mu|| > |\mu(G)|$ implies that $|\mu(f_{J_0})| > |\mu(f_\emptyset)|$ for some $J_0 \subset I \times \mathbb{N}$.

Let π_1 be the projection of $I \times \mathbb{N}$ onto the first coordinate. Then from case 1 we infer that there is a continuous character

$\alpha \in \mathrm{clo}[\{f_J \mid \pi_1(J) \subset \pi_1(J_0),\ J \subset I \times \mathbb{N}\}]$ such that $\mu(\alpha) = 0$. □

(3.24) QUESTION. Are (i), (ii), (iii), (iv) and (v) of (3.22) also equivalent when we assume that G is of type Z_p?

CHAPTER IV

GROUPS OF TYPE C IN THE CASE WHERE $\mathbb{Q}_p \subset K$

In this chapter we will assume that $\mathbb{Q}_p \subset K$.

§1. The idempotent elements of M(G).

In the classical case one can prove that when G is a locally compact group and $\mu \in M(G)$ is an idempotent, then there are compact subgroups $H_1,\ldots,H_n$ of G together with continuous characters $\alpha_1,\ldots,\alpha_n$ such that μ can be written as a linear combination of $\alpha_1 m_{H_1},\ldots,\alpha_n m_{H_n}$ [where m_{H_i} is the classical normalized Haar measure on H_i] [For example use [R-Chapter 3] and [G].

In the p-adic case, the only p-primary compact groups that have an Haar measure are the finite ones. This suggest the following conjecture.

(4.1) CONJECTURE. Let G be a p-primary torsional group. Then each idempotent element of M(G) has finite support.

In the case that G is of type Z_p, the norm on M(G) is multiplicative. Consequently, the only idempotents are the trivial ones and we see that the conjecture is true. Another case in which we can prove the conjecture is the case that G is isomorphic to C_p^I for some index set I. This will be done in [(4.2)-(4.15)].

In this section from now on G is a fixed group topologically isomorphic

<u>to</u> C_p^I <u>for some index set</u> I. P <u>denotes the set</u> $\{0,\dots,p-1\}$.
For the proof of the conjecture it is not a restriction, when we assume that K has p^{th} roots of unity. Therefore, <u>in this section from now on we assume that</u> K <u>has</u> p^{th} <u>roots of unity</u>.

The proofs of (4.2), (4.3), (4.4) and (4.5) are left to the reader.

(4.2) LEMMA. G^* is isomorphic to $\bigoplus_I C_p$. $\square$

(4.3) LEMMA. (a) Let H be a subgroup of G and let $g \in G$.
Then $\cup\{kg + H \mid k \in P\}$ is a subgroup of G.

(b) Let H^* be a subgroup of G^* and let $\rho \in G^*$.
Then $\cup\{\rho^k H^* \mid k \in P\}$ is a subgroup of G^*. $\square$

(4.4) LEMMA. (a) Let H_1 and H_2 be subgroups of G such that $H_2 \subset H_1$.
Then there is a subgroup H_3 of H_1 such that H_1 is the direct sum of H_2 and H_3.

(b) Let H_1^* and H_2^* be subgroups of G^* such that $H_2^* \subset H_1^*$.
Then there is a subgroup H_3^* of H_1^* such that H_1^* is the direct sum of H_2^* and H_3^*. $\square$

<u>From now on in this section</u> $\mu \in M(G)$ <u>will be a fixed idempotent measure</u>.
N <u>will be a fixed natural number such that</u> $||\mu|| \leq p^N$.

(4.5) LEMMA. $\hat{\mu}$ only takes the values zero and one. $\square$

(4.6) LEMMA. Let H be an open subgroup of G and let $\rho \in G^*$.
Then there is an $m \in \{0,\dots,p^N\}$ such that $\mu(\rho\zeta(H)) = m \cdot p^{-N}$.

<u>Proof</u>. From the formula $\zeta(H) = (\#\mathrm{Ann}\ H)^{-1}.\Sigma\{\gamma \mid \gamma \in \mathrm{Ann}\ H\}$ we infer that

$$\mu(\rho\zeta(H)) = (\#\mathrm{Ann}\ H)^{-1}.\ \Sigma\{\mu(\rho\gamma) \mid \gamma \in \mathrm{Ann}\ H\}.$$

By using Lemma (4.5) and the fact that $\#\{\rho\gamma \mid \gamma \in \mathrm{Ann}\ H\} = \#\{\gamma \mid \gamma \in \mathrm{Ann}\ H\}$, it follows that

$$\Sigma\{\mu(\rho\gamma) \mid \gamma \in \mathrm{Ann}\ H\} \in \{0,\ldots, \#\mathrm{Ann}\ H\}. \quad (i)$$

From the fact that $||\mu|| \geq |\mu(\rho\zeta(H))|$ it follows that

$$|\mu(\rho\zeta(H))| = |(\#\mathrm{Ann}\ H)^{-1}. \Sigma\{\mu(\rho\gamma) \mid \gamma \in \mathrm{Ann}(H)\}| \leq p^N \quad (ii).$$

Further $\#\mathrm{Ann}\ H$ is a power of p (iii).

Now combine (i), (ii) and (iii) to deduce that there is $m \in \{0,\ldots,p^N\}$ with $\mu(\rho\zeta(H)) = m.p^{-N}$. □

(4.7) COROLLARY. The set $\{\mu(\rho\zeta)) \mid H$ open subgroup of G, $\rho \in G^*\}$ is a finite set. □

For each $j \in \mathbb{N} \cup \{0\}$, P_j denotes the following assertion:

There is no sequence $G_1 \supset H_1 \supset G_2 \supset H_2 \supset \ldots$ of open subgroups of G such that $[G_n:H_n] = p$ $[n \in \mathbb{N}]$ and such that $\mu(G_n) - \mu(H_n) = j \cdot p^{-N}$ $[n \in \mathbb{N}]$.

(4.8) LEMMA. P_j is true for each $j \in \mathbb{N} \cup \{0\}$.

Proof. The proof in the case when $j > p^N$ follows immediately from (4.6). Let $i \in \{0,\ldots,p^N\}$ and suppose that P_i is false. Assume that P_j is true for each $j > i$. Then use Corollary (4.7) to deduce that we may assume that there is a $t \in \{0,\ldots,p^N\}$, $t > i$, together with a sequence $G_1 \supset H_1 \supset G_2 \supset H_2 \supset \ldots$ of open subgroups of G, with $[G_n:H_n] = p$, such that $\mu(G_n) = tp^{-N}$ and $\mu(G_n) - \mu(H_n) = i \cdot p^{-N}$.

For each $m > n$ let $\Lambda_{n,m}$ be a subgroup of $\mathrm{Ann}\ G_m$ such that $\mathrm{Ann}\ G_m$ is the direct sum of $\Lambda_{n,m}$ and $\mathrm{Ann}\ G_n$. Further, for each $n \in \mathbb{N}$ choose $\rho_n \in \mathrm{Ann}\ H_n \setminus \mathrm{Ann}\ G_n$.

By using (4.6) we see that there are two possibilities. Both of them will lead to a contradiction.

(i) There is a sequence $n(1) < m(1) < n(2) < m(2) < \dots$ of natural number together with $\alpha_k \in \Lambda_{n(k),m(k)}$ for each $k \in \mathbb{N}$ such that $\mu(\mathrm{Ann}\ \mathrm{U}\{(\alpha_k\ _{m(k)})^j.\ \gamma \mid j \in P,\ \gamma \in \mathrm{Ann}\ G_{n(k)}\}) \in \{(i+1)p^{-N},\dots,p^N.p^{-N}\}$.

(ii) The alternative to (i): there is an $M \in \mathbb{N}$ such that for each $m > M$ and for each $\alpha \in \Lambda_{M,m}$

$\mu(\mathrm{Ann}\ \mathrm{U}\{(\alpha\rho_m)^j\gamma \mid j \in P, \gamma \in \mathrm{Ann}\ G_M\}) \in \{0,p^{-N},\dots,i.p^{-N}\}$.

<u>Proof that</u> (i) <u>leads to a contradiction</u>.

Let $n(1) < m(1) < n(2) < m(2) < \dots$ be a sequence of natural numbers as above.

Denote $L_k \underset{D}{=} \mathrm{Ann}\ \mathrm{U}\{\alpha_k\rho_{m(k)})^j\gamma \mid j \in P, \gamma \in \mathrm{Ann}\ G_{n(k)}\}$.

From the finiteness of the set $\{(i+1)p^{-N},\dots,p^N.p^{-N}\}$ it follows that we may assume that there is an $s \subset \{i+1,\dots,p^N\}$ such that

$\mu(L_k) = s\ .\ p^{-N}$ for each $k \in \mathbb{N}$. (a)

It is almost trivial that $G_{n(k)} \supset L_k$ and that $[G_{n(k)} : L_k] = p$ for each $k \in \mathbb{N}$. (b)

From the facts that $n(k+1) > m(k)$, that $\alpha_k \in \mathrm{Ann}\ G_{m(k)}$ and that $\rho_{m(k)} \in \mathrm{Ann}\ H_{m(k)} \subset \mathrm{Ann}\ G_{m(k)+1} \subset \mathrm{Ann}\ G_{m(k+1)})$ it follows that $G_{n(k+1)} \subset L_k$. (c)

Combine (a), (b) and (c) to deduce a contradiction with the induction hypothesis.

<u>Proof that</u> (ii) <u>leads to a contradiction</u>.

We may assume that $M = 1$. Let $m > M$ and let $\alpha \in \Lambda_{1,m}$.

It follows that

$$(\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu((\alpha\rho_m)^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_1,\ j \in P \setminus \{0\}\} \in \{(pj-t).p^{-N} \mid j\in\{0,\dots,i\}\} \quad \text{(a)}.$$

and that

$$(\#\mathrm{Ann}\ G_m)^{-1}.\Sigma\{\mu(\rho_m^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_m,\ j \in P \setminus \{0\}\} = (pi-t).p^{-N} \quad \text{(b)}.$$

Further we have that

$$(\#\mathrm{Ann}\ G_m)^{-1}.\Sigma\{\mu(\rho_m^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_m,\ j \in P \setminus \{0\}\} =$$
$$(\#\mathrm{Ann}\ G_m)^{-1}.\Sigma\{\mu((\alpha\rho_m)^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_1,\ \alpha \in \Lambda_{1,m},\ j \in P \setminus \{0\}\} \quad \text{(c)}.$$

By combining (a), (b) and (c) and by using the fact that

$\#\mathrm{Ann}\ G_m = (\#\Lambda_{1,m}) \cdot (\#\mathrm{Ann}\ G_1)$ we deduce that for each $\alpha \in \Lambda_{1,m}$,

$$p^{-1}.(\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu((\alpha\rho_m)^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_1,\ j \in P\} = i \cdot p^{-N} \quad \text{(d)}$$

and $(\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu((\alpha\rho_m)^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_1,\ j \in P \setminus \{0\}\} = (pi-t)p^{-N}$ (e).

Now inductively choose elements $\beta_2 \in \Lambda_{1,2},\dots,\beta_m \in \Lambda_{1,m}$

such that $\#\{\llbracket \beta_2\rho_2,\dots,\beta_m\rho_m \rrbracket.\ \mathrm{Ann}\ G_1\} = p^{m-1}.\ (\#\mathrm{Ann}\ G_1)$.

Define L_m by $L_m \underset{D}{=} \mathrm{Ann}(\llbracket \beta_2\rho_2,\dots,\beta_m\rho_m \rrbracket.\mathrm{Ann}\ G_1)$.

Then from (d) and (e) it follows that

$$\mu(L_m) = p^{1-m}.(\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu((\beta_2\rho_2)^j\gamma) \mid \gamma \in \mathrm{Ann}\ G_1,\ j \in P\} +$$
$$p^{1-m}\ (\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu(\alpha\gamma) \mid \alpha \in \llbracket \beta_2\rho_2,\beta_3\rho_3 \rrbracket \setminus \llbracket \beta_2\rho_2 \rrbracket,\ \gamma \in \mathrm{Ann}\ G_1\} +$$
$$\dots + p^{1-m}.(\#\mathrm{Ann}\ G_1)^{-1}.\Sigma\{\mu(\alpha\gamma) \mid \alpha \in \llbracket \beta_2\rho_2,\dots,\beta_m\rho_m \rrbracket \setminus \llbracket \beta_2\rho_2,\dots,\beta_{m-1}\rho_{m-1} \rrbracket, \gamma\in\mathrm{Ann}\ G_1\}$$
$$= p^{1-m}\{p.i.p^{-N} + p.(pi-t)p^{-N} + \dots + p^{m-2}.\ (pi-t)p^{-N}\} =$$
$$p^{1-m}\{t \cdot p^{-N} + \{(pi-t)+p(pi-t)+\dots+p^{m-2}(pi-t)\}.p^{-N}\} = p^{1-m-N}.\{t+(pi-t).(1-p^{m-1})(1-p)^{-1}\}.$$

Consequently,

$$\mu(L_m) = |t(1-p) + (pi-t)(1-p^{m-1})|\ |p^{1-m-N}| = |p(-t+i)+p^{m-1}(t-pi)|.p^{m+N-1}.$$

$i \neq t$ and therefore $|\mu(L_m)|$ tends to infinity when m tends to infinity,

which gives a contradiction. □

(4.9) COROLLARY. Let $g \in G$. Then there is an open subgroup H of G, containing g, such that for each open subgroup L of H, containing g, we have $\mu(L) = \mu(H)$. □

(4.10) DEFINITION. Let $r \in \{\mu(H) \mid H \text{ is an open subgroup of } G\}$ be fixed. Put $S = \{(s_1,\ldots,s_{p-1}) \mid s_1,\ldots,s_{p-1} \in \{m.p^{-N} \mid m \in \mathbb{N}\cup\{0\}\}, s_1 \geq s_2 \geq \ldots \geq s_{p-1}, s_1 > s_{p-1},$

$$\Sigma\{s_i \mid i \in P \setminus \{0\}\} = r\}.$$

A total ordering "<" on s is defined as follows: $(s_1,\ldots,s_{p-1}) < (t_1,\ldots,t_{p-1})$ if the first non-zero number in the sequence $(s_1-t_1,\ldots,s_{p-1}-t_{p-1})$ is greater than zero.

Let H_0 be an open subgroup of G such that for each open subgroup $H \subset H_0$, $\mu(H) = \mu(H_0) = r$. Let

$\underline{s} = (s_1,\ldots,s_{p-1}) \in S$. We say that $\underline{s}$ appears in the sequence $H_0 \supset H_1 \supset \ldots$ on the place n [where $H_1, H_2, \ldots$ are open subgroups of G] if there is an element $\rho_n \in \mathrm{Ann}(H_{n+1}) \setminus \mathrm{Ann}(H_n)$ such that the sequence

$$\mu(\rho_n\zeta(H_n)),\ \mu(\rho_n^2\zeta(H_n)),\ldots,\mu(\rho_n^{p-1}\zeta(H_n))$$

is a permutation of the sequence $(s_1,\ldots,s_{p-1})$.

[(4.6) tells us, that for each $i \in P \setminus \{0\}$

$$\mu(\rho_n^i\zeta(H_n)) \in \{0,p^{-N},2p^{-N},\ldots,1\}].$$

We say that $(s_1,\ldots,s_{p-1})$ appears infinitely many times in the sequence $H_0 \supset H_1 \supset \ldots$ of open subgroups when $(s_1,\ldots,s_{p-1})$ appears on infinitely many places in this sequence. For $\underline{s} \in S$, the statement $Q_{\underline{s}}$ is the following.

> There is no sequence $H_0 \supset H_1 \supset H_2 \supset \ldots$ of open subgroups of G in which $\underline{s}$ appears infinitely often.

(4.11) LEMMA. Let everything be as in (4.9). Then $Q_{\underline{s}}$ is true for each $\underline{s} \in S$.

Proof. The proof in the cases where $\underline{s} < (r,0,\ldots,0)$ is trivial.

Let $t = (t_1,\dots,t_{p-1})$. Suppose that $Q_{\underline{t}}$ is false and assume that $Q_{\underline{s}}$ is true for each $\underline{s} < \underline{t}$. Then there is a sequence of open subgroups $H_0 \supset H_1 \supset H_2 \supset \dots$ of G together with $\rho_1,\rho_2,\dots \in G^*$, such that $\rho_n \in \mathrm{Ann}\, H_{n+1} \setminus \mathrm{Ann}\, H_n$ and such that the sequence

$$\mu(\rho_n\zeta(H_n)),\mu(\rho_n^2\zeta(H_n)),\dots,\mu(\rho_n^{p-1}\zeta(H_n))$$

is a permutation of the sequence $(t_1,\dots,t_{p-1})$ [each $n \in \mathbb{N}$].

Let for each $m > n$, $m,n \in \mathbb{N}$, $\Lambda_{n,m}$ be a subgroup of $\mathrm{Ann}\, H_m$ such that $\mathrm{Ann}\, H_m$ is the direct sum of $\Lambda_{n,m}$ and $\mathrm{Ann}\, H_n$ [see (4.4)].

Using the induction hypothesis, we first prove the following.

There is $M \in \mathbb{N}$ such that for each $n > M$ and each $\alpha_n \in \Lambda_{M,n}$ the sequence $\mu(\alpha_n.\rho_n.\zeta(H_M)),\mu(\alpha_n\rho_n^2.\zeta(H_M)),\dots,\mu(\alpha_n\rho_n^{p-1}.\zeta(H_M))$ is a permutation of the sequence $(t_1,t_2,\dots,t_{p-1})$ [see (a)].

Using (a) we will show that a contradiction follows [see (b)].

<u>Proof of (a)</u>

(a.1) We may assume that $\rho_1,\rho_2,\dots$ are such that for each $n \in \mathbb{N}$,

$$\mu(\rho_n.\zeta(H_n)) = t_1.$$

There are two possibilities

(1) We can find a sequence $n(1) < m(1) < n(2) < m(2) < \dots$ of natural numbers, together with $\alpha_i \in \Lambda_{n(i),m(i)}$ for each $i \in \mathbb{N}$, such that

$$\mu(\alpha_i\rho_{m(i)}\zeta(H_{n(i)})) \in \{t_1+p^{-N},t_1+2.p^{-N},\dots,1\}.$$

Using the finiteness of the set $\{t_1+p^{-N},t_1+2.p^{-N},\dots,1\}$ it follows that we may assume that $\mu(\alpha_i\rho_{m(i)}\zeta(H_{n(i)})) = q$ for some

$$q \in \{t_1+p^{-N},\ t_1+2.p^{-N},\dots,1\}.$$

Now use the finiteness of the set S, to infer that there is an $\underline{s} \in S$, $\underline{s} < \underline{t}$ such that $\underline{s}$ appears infinitely many times in the sequence $H_0 \supset H_{n(1)} \supset H_{m(1)+1} \supset H_{n(2)} \supset \dots$.

(2) There is an $M(1) \in \mathbb{N}$, such that for each $n > M(1)$ and each $\alpha_n \in \Lambda_{M(1),n}$

$$\mu(\alpha_n \rho_n \zeta(H_{M(1)})) \in \{0, p^{-N}, 2.p^{-N}, \dots, t_1\}.$$

Then for each $n \in \mathbb{N}$, $n > M(1)$ we have that

$t_1 = \mu(\rho_n \zeta(H_n)) = (\#\mathrm{Ann}\ H_n)^{-1} \Sigma\{\mu(\gamma) \mid \gamma \in \rho_n.\mathrm{Ann}\ H_n\} =$

$(\#\Lambda_{M(1),n})^{-1}.(\#\mathrm{Ann}\ H_{M(1)})^{-1}.\Sigma\{\mu(\alpha\gamma) \mid \alpha \in \Lambda_{M(1),n},\ \gamma \in \rho_n\ \mathrm{Ann}\ H_{M(1)}\}.$

It follows that $\mu(\alpha_n \rho_n \zeta(H_{M(1)})) = t_1$ for each $n \in \mathbb{N}$, $n > M(1)$ and $\alpha_n \in \Lambda_{M(1),n}$.

(a.ii) Let $k(1), k(2), \dots \in P \setminus \{0\}$ be such that $\mu(\rho_n^{k(n)} \zeta(H_n)) = t_2$. Then is an analogous way as we have done in (a.i), replacing ρ_n by $\rho_n^{k(n)}$, and by using the result of (a.i) we prove that there is $M(2) \in \mathbb{N}$, such that for each $n > M(2)$ and $\alpha_n \in \Lambda_{M(2),n}$, we have that

$$\mu(\alpha_n \rho_n^{k(n)} \zeta(H_{M(2)})) = t_2 .$$

Proceeding in this way, we deduce that there is $M \in \mathbb{N}$, such that for each $m > n > M$ and $\alpha \in \Lambda_{M,m}$ the sequence

$$\mu(\alpha\rho_m \zeta(H_M)), \dots, \mu(\alpha\rho_m^{p-1} \zeta(H_M))$$

is a permutation of the sequence $(t_1, \dots, t_{p-1})$.

(b) Now we show that (a) leads to a contradiction. We may assume that $M = 1$ [where M is as found in (a)].

(b.i) Let $n > 1$ be fixed. First we prove the following assertion.

There is $x_n \in H_1$ such that $\rho_1(x_n) \neq 1$ and $\rho_2(x_n) = \dots = \rho_n(x_n) = 1$.

Proof. Denote the restriction of ρ_i to H_1 by ρ_i' $(i \in \{1,\dots,n\})$. Then for each $i \in \{1,\dots,n\}$, ρ_i' is a continuous character on H_1. Some duality arguments show that to prove the assertion it is sufficient to prove that the subgroup of H_1^* generated by $\rho_1',\dots,\rho_n'$ is isomorphic to C_p^n. Hence, what we have to prove is that when $\prod_{i=1}^{n} (\rho_i(h))^{r_i} = 1$ for each $h \in H_1$, then r_i is a power of p for each $i \in \{1,\dots,n\}$. But this is easy. [For example the proof of the assertion that r_n is a power of p: First note that $\rho_i \in \mathrm{Ann}\, H_{i+1} \setminus \mathrm{Ann}\, H_i$. Therefore we can choose $h \in \mathrm{Ann}\,(H_n)$ such that $\rho_n(h) \neq 1$. Then $\prod_{i=1}^{n} (\rho_i(h))^{r_i} = (\rho_n(h))^{r_n}$ and it follows that r_n is a power of p.]

(b.ii) Denote $L_n = \mathrm{Ann}([\![\rho_1,\dots,\rho_n]\!].\ \mathrm{Ann}\, H_1)$. Choose $x_n \in H_1$ as announced in (b.i). Let $l \in P \setminus \{0\}$. Then

$\mu(lx_n+L_n) = (\#\mathrm{Ann}\, L_n)^{-1}.\Sigma\{\gamma(lx_n)\mu(\gamma) \mid \gamma \in \mathrm{Ann}\, L_n\} =$

$p^{-n}.(\#\mathrm{Ann}\, H_1)^{-1}.\Sigma\{(\alpha\gamma)(lx_n)\mu(\alpha\gamma) \mid \alpha \in [\![\rho_2,\dots,\rho_n]\!],\ \gamma \in [\![\rho_1]\!].\mathrm{Ann}\, H_1\}.$

Now let $\alpha \in [\![\rho_2,\dots,\rho_n]\!]$, $\alpha \neq 1$. Then

$\Sigma\ \{(\alpha\gamma)(lx_n)\mu(\alpha\gamma) \mid \gamma \in [\![\rho_1]\!]\ .\ \mathrm{Ann}\, H_1\} =$

$\quad \Sigma\{\gamma(lx_n)\mu(\alpha\gamma) \mid \gamma \in [\![\rho_1]\!]\ .\ \mathrm{Ann}\, H_1\} =$

$\quad \Sigma\{(\rho_1^k.\gamma)(lx_n)\mu(\alpha\rho_1^k\,\gamma) \mid \gamma \in \mathrm{Ann}\, H_1,\ k \in P\} =$

$\quad \Sigma\{\rho_1(x_n)^{k.l}\mu(\alpha\rho_1^k\gamma) \mid \gamma \in \mathrm{Ann}\, H_1,\ k \in P\}.$

Now let v be minimal such that $\alpha \in [\![\rho_2,\dots,\rho_v]\!]$. Let $r(v) \in \{1,\dots,p-1\}$ be such that $\alpha\rho_v^{-r(v)} \in [\![\rho_2,\dots,\rho_{v-1}]\!]$ and let $\beta_k = \alpha.(\rho_v)^{-r(v)}.(\rho_1)^k$ $[k \in P]$. Then it follows from (a) that for each $k \in P$,

$\Sigma\{\mu(\beta_k.\rho_v^{r(v)}\gamma) \mid \gamma \in \mathrm{Ann}\, H_1\} = \Sigma\{\mu(\rho_v^{r(v)}\gamma) \mid \gamma \in \mathrm{Ann}\, H_1\}.$

We deduce that

$\Sigma\{\rho_1(x_n)^{kl}.\mu(\alpha\rho_1^k\gamma) \mid \gamma \in \mathrm{Ann}\, H_1,\ k \in P\} =$

$(\Sigma\{(\rho_1(x_n))^{kl} \mid k \in P\})\ (\Sigma\{\mu(\alpha\gamma) \mid \gamma \in \text{Ann } H_1\})$.

But $\rho_1(x_n)$ is a non-trivial p^{th} root of unity and $l \in P \setminus \{0\}$. Therefore, $\Sigma\{\rho_1(x)^{kl} \mid k \in P\} = 0$.

It follows that

$\mu(lx_n+L_n) = p^{-n}.(\#\text{Ann } H_1)^{-1}.(\Sigma\{\gamma(lx_n)\mu(\gamma) \mid \gamma \in [\rho_1] . \text{Ann } H_1\}) =$

$p^{-n}.(\#\text{Ann } H_1)^{-1}.(\Sigma\{(\rho_1^k.\gamma)(lx_n)\mu(\rho_1^k\gamma) \mid \gamma \in \text{Ann } H_1,\ k \in P\}) =$

$p^{-n}.(\#\text{Ann } (H_1))^{-1}.(\Sigma\{\rho_1(klx_n)\mu(\rho_1^k\gamma) \mid \gamma \in \text{Ann } H_1,\ k \in P\}) =$

$p^{-n}.(\#\text{Ann } (H_1))^{-1}.(\Sigma\{\rho_1(klx_n)\mu(\rho_1^k\gamma) \mid \gamma \in \text{Ann } H_1,\ k \in P\}) =$

$p^{-n}.\Sigma\{\rho_1(klx_n)\mu(\rho_1^k\zeta(H_1) \mid k \in P\}$.

Now assume that there is $l_n \in P \setminus \{0\}$ such that $\mu(l_nx_n+L_n) \neq 0$. Using the fact that ρ_1 only takes finitely many values and that $||\mu|| \leq p^N$, it follows that there is a constant $c > 0$, independent of n such that

$$c \leq |\Sigma\{\rho_1(kl_nx_n)\mu(\rho_1^k\zeta(H_1) \mid k \in P\}|.$$

We infer that $|\mu(l_nx_n+L_n)| \geq c.p^n$. Now a contradiction follows easily from the fact that $||\mu|| \leq p^N$.

What is left is the proof of the assumption.

The only solution of the set of equations

$$\begin{aligned}
\rho_1(x_n)x_1+\ldots+(\rho_1(x_n))^{p-1}x_{p-1} &= -\mu(H_1)\\
\rho_1(2x_n)x_1+\ldots+(\rho_1(2x_n))^{p-1}x_{p-1} &= -\mu(H_1)\\
\vdots \qquad\qquad \vdots \qquad\qquad &\vdots\\
\rho_1((p-1)x_n)x_1+\ldots+(\rho_1((p-1)x_n))^{p-1}x_{p-1} &= -\mu(H_1)
\end{aligned}$$

is $(\mu(H_1),\ldots,\mu(H_1))$. The sequence $\mu(\rho_1\zeta(H_1)),\mu(\rho_1^2\zeta(H_1)),\ldots,\mu(\rho_1^{p-1}\zeta(H_1))$ is a permutation of the sequence $(t_1,\ldots,t_{p-1})$ and $t_1 > t_{p-1}$.

Consequently, there is an $l_n \in P \setminus \{0\}$ such that

$\rho_1(1_n x_n)\mu(\rho_1\zeta(H_1))+\rho_1(21_n x_n)\mu(\rho_1^2(\zeta(H_1))+\ldots\rho_1((p-1)1_n x_n)\mu(\rho_1^{p-1}\zeta(H_1))\neq-\mu(H_1)$.

Hence,

$$\Sigma\{\rho_1(k1_n x_n)\mu(\rho_1^k\zeta(H_1) \mid k \in P\} \neq 0. \quad \square$$

(4.12) COROLLARY. Let $g \in G$. Then there is an open subgroup H of G containing g such that for each open subgroup L of H containing g and each $\rho \in \text{Ann } L$, $\mu(\rho\zeta(H)) = \mu(H)$.

<u>Proof</u>. Suppose not. Choose H_0 as in (4.9). With induction we can choose open subgroups $H_0 \supset H_1 \supset H_2 \supset \ldots$ together with $\rho_n \in \text{Ann } H_n$ such that $\mu(\rho_n\zeta(H_n))\neq\mu(H_0)$ $[n\in\mathbb{N}]$. But then we get a contradiction with Lemma (4.11). $\square$

(4.13) LEMMA. Let $g \in G$. Then there is an open subgroup H of G containing g such that when L is an open subgroup of H containing g and when $h \in H\backslash L$, then $\mu(h+L) = 0$.

<u>Proof</u>. Choose H as in (4.12) and let L be an open subgroup of H containing g. Further let $h \in H\backslash L$. Choose $\rho \in \text{Ann } L \setminus \text{Ann } H$ such that $\rho(h) \neq 1$. Let Λ be such that Ann L is the direct sum of Λ and $[\rho]$. Ann H. Then

$\mu(h+L) = (\#\text{Ann } L)^{-1}.\Sigma\{\gamma(h)\mu(\gamma) \mid \gamma \in \text{Ann } L\} =$

$(\#\text{Ann } L)^{-1}.\Sigma\{(\alpha\rho^k\gamma)(h)\mu(\alpha\rho^k\gamma) \mid \gamma \in \text{Ann } H,\ k \in P,\ \alpha \in \Lambda\}$.

Now use corollary (4.11) and the fact that $h \in H$ to deduce that for each $\alpha \in \Lambda$

$(\#\text{Ann } L)^{-1}.\Sigma\{\alpha(h)\rho(kh)\gamma(h)\mu(\alpha\rho^k\gamma) \mid \gamma \in \text{Ann } H,\ k \in P\} =$

$p^{-1}.(\#\Lambda)^{-1}.(\#\text{Ann } H)^{-1}(\Sigma\{\rho(kh) \mid k \in P\}).(\Sigma\{\mu(\gamma) \mid \gamma \in \text{Ann } H\}) =$

$p^{-1}.(\#\Lambda)^{-1}.\alpha(h).\Sigma\{\rho(kh) \mid k \in P\}.\mu(H) = 0$.

Consequently, $\mu(h+L) = 0$. $\square$

(4.14) LEMMA. Let $g \in G$. Then there is an open subgroup H_g of G such that for each open subgroup $L \subset H_g$ and $h \in \llbracket g \rrbracket + H_g \setminus \llbracket g \rrbracket + L$ we have that $\mu(h+L) = 0$.

<u>Proof</u>. In the case when $g = 0$, we are done by Lemma (4.13). In the case that $g \neq 0$, let ρ be a K-valued continuous character with $\rho(g) \neq 1$. The kernel of ρ is denoted by H_0. For each $i \in P$ let $\mu_i \underset{D}{=} \rho^i.\mu$. Then it is clear that each μ_i is an idempotent. Use [4.13] to choose open subgroups $H_{1,i}$ and $H_{2,i}$ for each $i \in P$ such that

(i) $g \in H_{2,i}$

(ii) if $L \subset H_{1,i}$ and $h \in H_{1,i} \setminus L$, then $\mu_i(h+L) = 0$ [L an open subgroup of G]

(iii) if $g \in L$, $L \subset H_{2,i}$ and $h \in H_{2,i} \setminus L$ then $\mu_i(h+L) = 0$ [L an open subgroup of G].

Define H_g by the formula

$$H_g = H_0 \cap \bigcap_{i=0}^{p-1} H_{1,i} \cap \bigcap_{i=0}^{p-1} H_{2,i}.$$

Now choose $L \subset H_g$ and let $h \in \llbracket g \rrbracket + H_g \setminus \llbracket g \rrbracket + L$. Let $k \in H_g$ be such that $h \in k + \llbracket g \rrbracket$. Then for each $i \in P$

$$0 = \mu_i(h + \llbracket g \rrbracket + L) = \mu_i(k + \llbracket g \rrbracket + L) =$$

$$\Sigma\{\mu_i(\{rg + H_0\} \cap \{k + \llbracket g \rrbracket + L\}) \mid r \subset P\} =$$

$$\Sigma\{\rho(irg)\mu(\{rg+H_0\} \cap \{k + \llbracket g \rrbracket + L\}) \mid r \in P\} =$$

$$\Sigma\{\rho(irg)\mu(\{rg + H_0\} \cap \{k + sg + L\}) \mid r \in P,\ s \in P\}.$$

Now using the fact that $1 + L \subset H_0$, we infer that

$$0 = \Sigma\{\rho(irg)\mu(rg + 1 + L) \mid r \in P\}.$$

We already know that $\mu(1 + L) = 0$. It is an easy exercise to show that the matrix $(\rho(irg))_{i,r\in P}$ is invertible.
Consequently, $\mu(rg + 1 + L) = 0$ for each $r \in P$ and in particular

$\mu(h + L) = 0$. $\square$

(4.15) THEOREM. $\mathrm{supp}(\mu)$ is a finite set.

Proof. For each $g \in G$ choose H_g as in (4.13). It is clear that the set $\{[\![g]\!] + H_g \mid g \in G\}$ is an open covering of G. Therefore, by the compactness of G, there are $g_1,\dots,g_n \in G$ such that

$$G = \bigcup \{[\![g_i]\!] + H_{g_i} \mid i \in \{1,\dots,n\}\}.$$

I am going to prove that $\mathrm{supp}\ \mu \subset [\![g_1,\dots,g_n]\!]$ and consequently that $\mathrm{supp}\ \mu$ is a finite set.

To prove the assertion, it suffices to prove that $\mu(U) = 0$ for every clopen subset U of G such that $U \cap [\![g_1,\dots,g_n]\!] = \emptyset$. Each clopen subset is a finite union of cosets of an open subgroup of $H_{g_1} \cap \dots \cap H_{g_n}$. We see that we are done if we can prove that $\mu(h + H) = 0$, when H is an open subgroup of G contained in $\bigcap \{H_{g_i} \mid i \in \{1,\dots,n\}\}$ and h is such that $(h + H) \cap [\![g_1,\dots,g_n]\!] = \emptyset$. Let g_i be such that $h \in [\![g_i]\!] + H_{g_i}$. From the fact that $(h + H) \cap [\![g_1,\dots,g_n]\!] = \emptyset$ it follows that $h \notin [\![g_i]\!]+H$. Consequently, $h \in [\![g_i]\!] + H_{g_i} \setminus [\![g_i]\!] + H$.
Now use (4.13) and we are done. $\square$

The proof of (4.16) is left to the reader.

(4.16) COROLLARY. Let $G \cong (C_p)^I$ for some index set I and let $\nu \in M(G)$. Then the following statements are equivalent.

(i) ν is an idempotent;

(ii) there is a finite subgroup H of G together with $\alpha_1,\ldots,\alpha_n \in G^*$, such that $\nu = (\alpha_1+\ldots+\alpha_n)m_H$, where m_H is the normalised Haar measure on H;

(iii) there is a subgroup Λ of G^* with $[G^*:\Lambda]$ is finite, together with $\beta_1,\ldots,\beta_n \in G^*$ such that

$$\hat{\nu} = \zeta(\cup\{\beta_i\Lambda \mid i \in \{1,\ldots,n\}\}). \qquad \square$$

§2. Some general remarks.

In this section we assume the following. Let G be a compact p-primary group and $g \in G$ have finite order p^n. Then all p^n th roots of unity are in K.

(4.17) LEMMA. Let G be an infinite group of type C. Let $\mu \in M(G)$ and $\varepsilon > 0$. Then the set

$$\{\gamma \mid \gamma \in G^*, \quad ||\hat{\mu}|| - |\hat{\mu}(\gamma)| < \varepsilon\}$$

is infinite.

Proof. We may assume that $||\hat{\mu}|| - |\hat{\mu}(1)| < \varepsilon$. Suppose that $\gamma_1,\ldots,\gamma_n \in G^*$ are such that

$$||\hat{\mu}|| - |\hat{\mu}(\gamma)| \geq \varepsilon \quad \text{for all } \gamma \in G^* \setminus \{\gamma_1,\ldots,\gamma_n\}.$$

Denote $\Lambda = [\![\gamma_1,\ldots,\gamma_n]\!]$. G is of type C and K has "enough" roots of unity. Hence, $G^* \cong \bigoplus_{j\in J} C_{p^{n(j)}}$ for some infinite index set J and some suitable family of natural numbers $(n(j))_{j\in J}$. The set Λ is finite. Now it follows easily that there is a sequence $\Lambda_1,\Lambda_2,\ldots$ of finite subgroups of G^* such that $\Lambda_i \cap \Lambda = \{1\}$ and $\lim_{i\to\infty}(\#\Lambda_i) = \infty$. For each $i \in \mathbb{N}$, $\#\Lambda_i$ is a power of p. Therefore,

$\lim_{i\to\infty} |\mu(\mathrm{Ann}\ \Lambda_i)| = \lim_{i\to\infty} |\ (\#\Lambda_i)^{-1}|\ |\Sigma\{\mu(\gamma)|\ \gamma \in \Lambda_i\}| =$

$\lim_{i\to\infty} |(\#\Lambda_i)^{-1}|\ |\mu(1)| = \infty$, which is a contradiction. □

(4.18) THEOREM. Let G be an infinite group such that the elements with finite order are dense in G. Let $\mu \in M(G)$ and let $\varepsilon > 0$. Then

$$\#\{\gamma \mid \gamma \in G^*, ||\hat{\mu}|| - |\hat{\mu}(\gamma)| < \varepsilon\}$$

is an infinite set.

Proof. There is a closed subgroup H of G such that G/H is an infinite group and such that G/H is of type C [use (2.19)]. Then it follows from (4.17) that

$$\#\{\gamma \mid \gamma \in \mathrm{Ann}\ H, ||\mu|| - |\hat{\mu}(\gamma)| < \varepsilon$$

is an infinite and the proof is done. □

(4.19) COROLLARY. Let G be as in (4.18). Let $\mu \in M(G)$, $\mu \neq 0$. Then $\hat{\mu} \notin c_\infty(G^*)$. □

In the case that K is locally compact, one can sharpen (4.17) and (4.18).

(4.20) THEOREM. Assume that K is locally compact and let $\mu \in M(G)$. Let $\gamma_0 \in G^*$ and let $\varepsilon > 0$. Then the set

$$\{\gamma \mid \gamma \in G^*, |\hat{\mu}(\gamma) - \hat{\mu}(\gamma_0)\ | < \varepsilon\}$$

is an infinite set.

Proof. Denote $V_1 = \{\lambda \mid \lambda \in K, |\lambda - \hat{\mu}(\gamma_0)\ | < \varepsilon\}$ and

$$V_2 = \{\lambda \mid \lambda \in K, |\lambda - \hat{\mu}(\gamma_0)| \geq \varepsilon, |\lambda| \leq ||\hat{\mu}||\}.$$

Then V_1 and V_2 are clopen subsets of the compact set

$\{\lambda \mid \lambda \in K, |\lambda| \leq ||\hat{\mu}||\}$.

Now use [v.R.-(5.28)] to deduce that there is a polynomial P such that $P(V_1) \subset B_{\frac{1}{2}}(1)$ and $P(V_2) \subset B_{\frac{1}{2}}(0)$.

Let $\nu \underset{D}{=} P\mu$ [i.e. $\nu = a_0\bar{0} + a_1\mu + \ldots + a_n\mu^n$, when $P = a_0+a_1X+\ldots+a_nX^n$].

Suppose that the set $\{\gamma \mid \gamma \in G^*, |\hat{\mu}(\gamma) - \hat{\mu}(\gamma_0)| < \varepsilon\}$ is finite. Then we see that the set

$\{\gamma \mid \gamma \in G^*, ||\hat{\nu}|| - |\hat{\nu}(\gamma)| < \frac{1}{2}\}$

is finite, which is a contradiction with (4.18). $\square$

(4.21) REMARK. The condition that K is locally compact together with the fact that K has "enough" roots of unity, implies that our group G in (4.20) must be of bounded order. Hence, $G \cong \prod_{j\in J} C_{p^{n(j)}}$ for some index set J and some bounded subset

$\{n(j) \mid j \in J\}$ of $\mathbb{N}$. $\square$

(4.22) REMARK. The following counterexample shows that the assumption "K is locally compact" in (4.20) is not redundant. Let K be not locally compact and let $G = (C_p)^{\mathbb{N}}$.

For each $i \in \mathbb{N}$, $a_i \in G$ is defined by

$a_i(j) = 0$ when $j \neq i$

$a_i(j) = 1$ when $j = i$.

Let $1 > c > 0$ be a constant. Choose a sequence $\lambda_1, \lambda_2, \ldots \in K$ such that $c < |\sum_{j\in J} \lambda_j| \leq 1$ for each finite subset J of $\mathbb{N}$ [see v.R.-(1.B)].

Let $\lambda_0 \in K$ be such that $|\lambda_0| < p^{-1}.c$. It is easy to see that the formula

$$f \to \lambda_0 f(0) + \Sigma\{\lambda_j((p-1)f(0) - f(a_j) - \ldots - f((p-1)a_j)) \mid j \in \mathbb{N}\}$$

defines a continuous functional on C(G). Denote the measure corresponding to this functional with μ. Then for each $\gamma \in G^*$, $\gamma \neq 1$, we have that

$|\hat{\mu}(\gamma)\ | > p^{-1}.c$ but $|\hat{\mu}(1)| = |\lambda_0| < p^{-1}.c$. Hence, the set

$$\{\gamma \mid |\hat{\mu}(\gamma) - \hat{\mu}(1)| < p^{-1}.c\}$$

is finite. □

(4.23) REMARK. One might think that one can prove the equivalence of the analogues of point (i) and (v) in theorem (3.22). However, this is not true, at least not in the case when K is not locally compact. As an example let μ be as constructed in (4.22). Assume that $\lambda_0 \neq 0$. Then clearly $\mu(\alpha) \neq 0$ for each $\alpha \in G^*$. Suppose that μ has an inverse ν.

Then for each $\alpha \in G^*$, $\alpha \neq 1$, $|\nu(\alpha)| = |\mu(\alpha)|^{-1} < p.c^{-1}$ and $|\nu(1)| = |\mu(1)|^{-1} > p.c^{-1}$. But then we have a contradiction with (4.17). □

(4.24) QUESTION. Let K be locally compact. Assume that G is compatible with K. Does $|\mu(\alpha)| \neq 0$ for each $\alpha \in G^*$, imply that μ is invertible? □

(4.25) REMARK. It is in general not true that each homomorphism $M(G) \to K$ is of type $\mu \to \mu(\alpha)$ for some fixed $\alpha \in G^*$ [$\mu \in M(G)$] as the following counterexample shows.

Assume that $p > 2$ [this assumption is not essential!]. Let $G = (C_p)^{\mathbb{N}}$. Let $\Lambda_1,\ldots,\Lambda_n$ be subgroups of G^*, such that $[G : \Lambda_i]$ is finite for each $i \in \{1,\ldots,n\}$. Then $[G^* : (\Lambda_1 \cap \ldots \cap \Lambda_n)]$ is finite. Consequently, there is a filter $\mathcal{B}$ on G^* generated by the subgroups of G^* that have finite index.

Then from (4.16) it follows that for each idempotent $\mu \in M(G)$ with $\mu(1) = 0$, $B = \{\gamma \mid \gamma \in G^*, \hat{\mu}(\gamma) = 0\} \in \mathcal{B}$.

For each $i \in \mathbb{N}$, $a_i \in G$ is defined by

$a_i(j) = 0$ when $i \neq j$,

$a_i(j) = 1$ when $i = j$.

For each $i \in \mathbb{N}$, define $\alpha_i \in G^*$ by

$\alpha_i(a_j) = 1$ when $i \neq j$,

$\alpha_i(a_j) = \pi$ when $i = j$

where π is a p^{th} root of unity.

Put $A = \{\alpha_i \mid i \in \mathbb{N}\}$. Then $1 \notin A$ and therefore surely not $A \in \mathcal{B}$. Suppose there is a subgroup Λ of G^* that has finite index and that is contained in $G^* \backslash A$. There is $N \in \mathbb{N}$ such that $a_n \in \Lambda$ when $n > N$ and a contradiction follows. Consequently, $A \notin \mathcal{B}$ and $G^* \backslash A \notin \mathcal{B}$ and therefore $\mathcal{B}$ is not maximal.

Let $\mathcal{M}$ be an ultrafilter on G^* such that $A \in \mathcal{M}$ and such that $\mathcal{B} \subset \mathcal{M}$. Then $\mathcal{M}$ is free. Let $\Psi : l^\infty(\Gamma) \to K$ be the homomorphism corresponding to $\mathcal{M}$ [i.e. $\Psi((a_\gamma)_{\gamma\in\Gamma})$ is the unique element of

$$\bigcap_{B\in\mathcal{M}} \mathrm{clo}\{a_\gamma \mid \gamma \in B\}] .$$

Define $\Phi : M(G) \to K$ by

$$\Phi(\mu) = \Psi(\hat{\mu}) \quad [\mu \in M(G)].$$

Let $\beta \in G^*$, $\beta \neq 1$ then there is an idempotent element $\nu \in M(G)$ such that $\nu(1) = 0$ and $\nu(\beta) = 1$. Then

$$\Phi(\nu) = \Psi(\hat{\nu}) = 0 \neq \nu(\beta). \qquad \text{(a)}$$

Now define the measure ρ on G as follows. Let m be the normalised K-valued Haar measure on $(C_2)^{\mathbb{N}}$. $\phi : (C_2)^{\mathbb{N}} \to (C_p)^{\mathbb{N}}$ is defined by

$$\phi((a_n)_{n\in\mathbb{N}}) = (a_n)_{n\in\mathbb{N}} \qquad [a_n \in \{0,1\} \text{ for each } n \in \mathbb{N}].$$

Let $\rho(f) \underset{D}{=} m(f\circ\phi) \quad [f \in C(G)]$.

For each $\alpha_n \in A$ we have that $\rho(\alpha_n) = m(\alpha_n o \phi) = \frac{1}{2} + \frac{1}{2}\pi$. We infer that

$$\Phi(\rho) = \Psi(\hat{\rho}) = \tfrac{1}{2} + \tfrac{1}{2}\pi \neq 1 = \hat{\delta}(1) \qquad \text{(b)}$$

By combining (a) and (b) we deduce that Φ is not of type $\mu \to \mu(\alpha)$ for some $\alpha \in G^*$ $[\mu \in M(G)]$. $\square$

(4.26) QUESTION. What can be said about the homomorphisms $M(G) \to K$ in the case where K is not locally compact? $\square$

CHAPTER V

p-PRIMARY GROUPS IN THE CASE WHERE $\mathbb{F}_p \subset K$

In this chapter G will always stand for a p-primary group. We will always assume that $\mathbb{F}_p \subset K$.

§1. General results.

(5.1) LEMMA. Let G be finite. Let $N \in \mathbb{N}$ be such that $p^N g = 0$ for each $g \in G$. Then $\mu^{p^N} = (\mu(G))^{p^N} . \bar{0}$ for each $\mu \in M(G)$.

Proof. $\mathbb{F}_p \subset K$ and accordingly, $(\sum_{i=1}^{n} \{\mu(s).\bar{s} | s \in H\})^{p^N} = \Sigma\{(\mu(s)\ \bar{s})^{p^N} \mid s \in H\} = (\Sigma\{\mu(s) \mid s \in H\})^{p^N} . \bar{0} = (\mu(G))^{p^N} . \bar{0}$. □

(5.2) is a generalization of (5.1).

(5.2) THEOREM. Let G be compact and let H be an open subgroup of G. $N \in \mathbb{N}$ is such that $p^N g \in H$ for each $g \in G$. Then $\mu^{p^N}(H) = (\mu(G))^{p^N}$ and $(\mu^{p^N})(h+H) = 0$ when $h \notin H$.

Proof. Denote the natural homomorphism $G \to G/H$ by ϕ. $\Phi : M(G) \to M(G/H)$ is the homomorphism induced by ϕ. Then from (5.1) we infer that $\Phi(\mu^{p^N}) = (\Phi(\mu))^{p^N} = ((\Phi(\mu))(G/H))^{p^N} . \overline{\phi(0)} = (\mu(G))^{p^N} . \overline{\phi(0)}$. Now the theorem follows at once. □

In a zerodimensional compact group each clopen set is a finite union of cosets of an open subgroup. Hence,

(5.3) COROLLARY. Let G be compact and let $U \subset G$ be a clopen set. Let $\mu \in M(G)$. Then there is $N \in \mathbb{N}$ such that $\mu^{p^N}(U) = 0$ if $0 \notin U$ and $\mu^{p^N}(U) = (\mu(G))^{p^N}$ if $0 \in U$. □

Now we can prove (3.18) and (3.19) in the case that characteristic K is finite.

(5.4) THEOREM [compare (3.19)]. Let $\mu \in M(G)$. Assume that $||\mu|| = |\mu(G)| \neq 0$. Then μ is invertible and $||\mu^{-1}|| = |\mu(G)|^{-1}$.

<u>Proof</u>. Again we may assume that G is compact and that $\mu(G) = 1$ [see (3.19)]. Let H be a clopen subgroup of G and $\Phi : M(G) \to M(G/H)$ the canonical homomorphism. Then $(\Phi(\mu))(G/H) = 1$ and therefore, by using (5.1), it follows that there is $N \in \mathbb{N}$ such that $\Phi(\mu^{p^N}) = \Phi(\bar{0})$ for each $n \geq N$. Hence, for each $n \geq N$, $\Phi(\mu^{p^n-1})$ is the inverse of $\Phi(\mu)$. We deduce that for $n,m \geq N$, $\Phi(\mu^{p^n-1}) = \Phi(\mu^{p^m-1})$. Consequently, let $U \subset G$ be a clopen set, then $\lim_{n\to\infty} \mu^{p^n-1}(U)$ exists.
It is clear that for each clopen set $U \subset G$, $|(\mu^{p^n-1})(U)| \leq 1$. Now define $\nu \in M(G)$ by the formula $\nu(U) = \lim_{n\to\infty} \mu^{p^n-1}(U)$ [U a clopen set of G].
Then it is easy to show that ν is the inverse of μ. It is almost trivial to show that $||\nu|| = |\mu(G)|^{-1}$. □

For the following theorem, compare (4.1).

(5.5) THEOREM. The only idempotents of M(G) are the trivial ones [i.e. 0 and $\bar{0}$].

Proof. Let $\mu \in M(G)$ be an idempotent and let $1 > \varepsilon > 0$. There is $\nu \in M(G)$, having compact support, such that $||\mu - \nu|| \leq \varepsilon$. Let $U \subset G$ be a clopen set such that $0 \notin U$. From (5.3) we infer that we can choose an $n \in \mathbb{N}$ with $\nu^{p^n}(U) = 0$. Consequently,

$$|\mu(U)| = |\mu^{p^n}(U)| = |\mu^{p^n}(U) - \nu^{p^n}(U)| \leq ||\mu^{p^n} - \nu^{p^n}||$$
$$\leq ||\mu - \nu||^{p^n} \leq \varepsilon.$$

By letting ε tend to 0 it follows that $\mu(U) = 0$. Hence, supp $\mu = \{0\}$. The rest of the proof is trivial. □

(5.6) follows immediately from (5.2) and (5.3).

(5.6) COROLLARY. Let G be compact and of bounded order. Let $N \in \mathbb{N}$ be such that $p^N g = 0$ for each $g \in G$. Then $\mu^{p^N} = (\mu(G))^{p^N} . \bar{0}$ for each $\mu \in M(G)$. □

(5.7) and (5.8) are consequences of (5.6).

(5.7) COROLLARY. Let G be a compact group of bounded order. Let $\mu \in M(G)$. Then the following two statements are equivalent:

(i) μ is invertible;

(ii) $\mu(G) \neq 0$. □

(5.8) COROLLARY. Let G be compact and of bounded order. Then $M = \{\mu \mid \mu \in M(G), \mu(G) = 0\}$ is the only maximal ideal in $M(G)$. □

(5.9) REMARK. There are no analogues of (5.7) and (5.8) in the case thet G is not of bounded order, even when G is of type C.

Counterexample. Let $G = \prod_{n \in \mathbb{N}} C_{p^n}$. Denote $a = (1,1,1,\ldots)$. Choose $\alpha, \beta \in K$

such that $|\alpha| = |\beta| > 1$ and such that $\alpha + \beta = 1$. Define $\mu \in M(G)$ by $\mu = \alpha.\bar{0} + \beta.\bar{a}$. For each $n \in \mathbb{N}$, H_n is the subgroup of G defined by $H_n = \{x \mid x \in G, x_1=\ldots=x_n = 0\}$. Clearly $\mu(G) = 1$. Now assume that μ is invertible. Denote its inverse by ν. For each $n \in \mathbb{N}$, we have that $\mu^{p^n}(H) = 1$ and $\mu^{p^n}(h+H) = 0$ as soon as $h + H$ is a coset of H with $h + H \neq H$. It follows that necessarily $\nu(H_n) = \mu^{p^n-1}(H_n)$. But

$$|\mu^{p^n-1}(H_n)| = |(\alpha\bar{0} + \beta\bar{a})^{p^n-1}(H_n)| = |\alpha|^{p^n-1}.$$

Then a contradiction follows with the fact that the set $\{\nu(H_n) \mid n \in \mathbb{N}\}$ is bounded. Now it is also clear that $\{\mu \mid \mu \subset M(G), \mu(G) = 0\}$ is not the only maximal ideal in M(G). □

(5.10) DEFINITION. Let E be a vector space and let D be a linear subspace of E. Let $U : G \to L(E,E)$ be a representation. Then D is called <u>shift invariant</u> if $U_x d \in D$ for each $x \in G$. In the case that $E = C(G)$, D is called shift invariant if $\bar{x} * f \in D$ for each $f \in D$ and each $x \in G$. □

Another consequence of (5.6) is the following.

(5.11) THEOREM. Let G be compact and of bounded order. Let V be a closed, non-trivial, shift invariant subspace of C(G). Then $1 \in V$.

<u>Proof.</u> It is no restriction to assume that $V = \mathrm{clo}[\![\{\bar{s}*f \mid s \in G\}]\!]$ for some $f \in C(G)$. For the moment let us assume the following.

For each $g \in C(G)$, $g \notin V$, there is $\mu \in M(G)$ such that $\mu(G) \neq 0$ and such that $\mu(h) = 0$ for each $h \in V$.

Now suppose that $1 \notin V$. Let $\rho \in M(G)$ be such that $\rho(1) \neq 0$ and $\rho(h) = 0$ for each $h \in V$. From (5.7) we infer that ρ is invertible. Consequently, for each $s \subset G$, $f(s) = (\rho^{-1}*\rho)(\bar{s}*f) = 0$, which gives a contradiction with the fact that V is non-trivial.

Proof of the assumption. In the case that K is spherically complete, this is an application of the Hahn-Banach theorem. For the general case, note that the map $s \to \bar{s} * f$ is continuous. Hence, the set $\{\bar{s} * f \mid s \in G\}$ is compact in C(G). It follows that V is of countable type. From [v.R.-(3.18)] and [v.R.-(5.22)] we deduce that there is a closed linear subspace V' of C(G) such that C(G) is the [topological] direct sum of V and V'. V' has an orthonormal base see [v.R.-5.9] and [v.R.-(5.22)]. Let $(e_i)_{i \in I}$ be such a base. $g \notin V$ and therefore its component in V' is non-trivial. We infer that there is $i \in I$ such that the projection of g onto $[e_i]$ is non-trivial. Now the assumption follows at once. $\square$

(5.12) REMARK. Again there is no analogue of (5.11), even in the case that G is of type C, when G is not of bounded order.

Counterexample. Let $G = C_p \times C_{p^2} \times \ldots$. Denote $a = (1,1,1,\ldots)$. Choose $\alpha \in K$ such that $\alpha \neq 1$ and such that $|\alpha - 1| < 1$. For the moment make the following assumption.

There is a function $f \in C(G)$, $f \neq 0$, such that $f(a+x) = \alpha f(x)$ for each $x \in G$.

Then define μ by $\mu = \alpha\bar{0} - \bar{a}$. Then for each $s \in G$, $\mu(\bar{s} * f) = \alpha f(s) - f(a+s) = \alpha f(s) - \alpha f(s) = 0$. But $\mu(1) = \alpha - 1 \neq 0$. Therefore, $1 \notin \text{clo}\,[\{\bar{s} * f \mid s \in G\}]$.

Construction of f.

Put $G_0 = \{x \mid x \in G$, there is $N \in \mathbb{N}$ such that $x(n) = 0$ for each $n > N\}$. We are going to define f on G_0. Let $l, k \in \mathbb{N} \cup \{0\}$. Then by $[l - k]_n$ we mean the element of $\{0,\ldots,p^n-1\}$ which, modulo p^n, is equivalent to

$1 - k$. For each $x \in G_0$, $x = (x(1), x(2), \ldots), x(i) \in \{0, \ldots, p^i - 1\}$, define $f(x)$ by the formula

$$f(x) = \alpha^{\Sigma\{[x(i)-x(i+1)]_i \mid i \in \mathbb{N}\}}.$$

Now let $x = (x(1), \ldots, x(n), 0, 0, \ldots)$ and $x' = (x(1), \ldots, x(n), x(n+1), 0, \ldots)$.

Then $|f(x')/f(x) - 1| = |\alpha^{x(n+1)-x(n)+[x(n)-x(n+1)]_n} - 1|$.

By using the fact that $\lim_{n\to\infty} \alpha^{p^n} = 1$ it follows that f is uniformly continuous on G_0. Define f on the whole of G by continuity. What is left to prove is that $f(a+x) = \alpha f(x)$ for each $x \in G$. For this, it suffices to prove that $f(a+x) = \alpha f(x)$ for each $x \in G_0$. Let $x(1), \ldots, x(n), 0, \ldots)$. Denote $a_1 = (1,0,0,\ldots)$, $a_2 = (1,1,0,\ldots)$, etc. Then $f(a+x) = \lim_{m\to\infty} f(a_m+x)$. But for each $m > n$, $f(a_m+x) = \alpha f(x)$ and we are done. □

We can prove an analogue of (5.11) for a larger class of groups in the case that V is a finite dimensional, shift invariant subspace of $C(G)$. First we need a lemma, that is interesting in its own right.

(5.13) LEMMA. Let G be compact. Assume that the elements that have finite order are dense in G. Let $f \in C(G)$, $f \neq 0$, be such that the linear space $V = [\![\{\bar{s} * f \mid s \in G\}]\!]$ is finite dimensional. Let $\overline{s(1)} * f, \ldots, \overline{s(n)} * f$ be a base of V. Choose $s \in G$ and let $\lambda(1), \ldots, \lambda(n)$ be such that $\bar{s} * f = \sum_{i=1}^{n} \lambda(i)\, \overline{s(i)} * f$. Then $\sum_{i=1}^{n} \lambda(i) = 1$.

Proof. It is of course no restriction when we assume that $\|f\| = 1$. First the lemma is proved in the case that $s, s(1), \ldots, s(n)$ all have finite order. Then there is a finite subgroup H of G such that $s, s(1), \ldots, s(n) \in H$. The element

$\bar{s} - \sum_{i=1}^{n} \lambda(i)\overline{s(i)}$ is not invertible in M(H). [$(\bar{s} - \sum_{i=1}^{n} \lambda(i)\overline{s(i)}) * f = 0$ and f is non-trivial]. Now use (5.7) to infer that

$$(\bar{s} - \sum_{i=1}^{n} \lambda(i)\overline{s(i)})(H) = 0.$$

Consequently, $\sum_{i=1}^{n} \lambda(i) = 1$.

For the general case, note that there is $0 < r \leq 1$ such that $\|\sum_{i=1}^{n} \alpha(i)\overline{s(i)} * f\| \geq r \,.\, \max |\alpha(i)|$ [see [v.R.-(3.15)]]. Choose $r > \varepsilon > 0$. The map $x \to \bar{x} * f$ is continuous and the elements of G that have finite order are dense in G. It follows that we can choose t, t(1),...,t(n) $\in$ G that have finite order and such that $\|\bar{s} * f - \bar{t} * f\| < \varepsilon$ and $\|\overline{s(i)} * f - \overline{t(i)} * f\| < \varepsilon$ [each $i \in \{1,\dots,n\}$]. The set $\overline{t(1)} * f,\dots,\overline{t(n)} * f$ is a base of V. [Let

$$\sum_{i=1}^{n} \alpha(i)\overline{t(i)} * f = 0.$$

Then

$$\varepsilon.\max|\alpha(i)| \geq \|\sum_{i=1}^{n} \alpha(i)(\overline{s(i)}*f-\overline{t(i)}*f)\| \geq r.\max|\alpha(i)|$$

and we deduce that $\alpha(i) = 0$ for each $i \in \{1,\dots,n\}$]. Let $\rho(1),\dots,\rho(n)$ be such that

$$\bar{t} * f = \sum_{i=1}^{n} \rho(i)\overline{t(i)} * f.$$

By using some standard arguments we infer that

$$|\sum_{i=1}^{n} (\rho(i) - \lambda(i))| \leq \varepsilon \,.\, r^{-2}.$$

Hence, by using the first part of this proof, we deduce that

$$|1 - \sum_{i=1}^{n} \lambda(i)| \leq \varepsilon \,.\, r^{-2}.$$

Now let ε tend to 0. □

(5.14) THEOREM. Let G be a compact group such that the elements with finite order are dense in G. Let V be a finite dimensional, shift invariant linear subspace of $C(G)$. Assume that $V \neq \{0\}$. Then $1 \in V$.

Proof. Again, it is no restriction when we assume that

$$V = [\![\{\bar{s} * f \mid s \in G\}]\!] \text{ for some } f \in C(G).$$

Further we may of course assume that f is such that $f(0) \neq 0$. From the proof of (5.13) it follows that we can choose a base $\overline{s(1)} * f,\ldots,\overline{s(n)} * f$ of V such that $s(i)$ has finite order for each $i \in \{1,\ldots,n\}$. From (5.11) we deduce that there are $\lambda(1),\ldots,\lambda(n) \in K$ such that

$$\sum_{i=1}^{n} \lambda(i)(\overline{s(i)} * f)(x) = 1 \text{ for each } x \in [\![s(1),\ldots,s(n)]\!].$$

Now let $s \in G$ be arbitrary. There are $\alpha(1),\ldots,\alpha(n) \in K$ such that $\sum_{i=1}^{n} \alpha(i) = 1$ and $\bar{s} * f = \sum_{i=1}^{n} \alpha(i)\overline{s(i)} * f$ [see (5.13)]. We infer that

$$\sum_{i=1}^{n} \lambda(i)(\overline{s(i)} * f)(s) = \sum_{i=1}^{n} \lambda(i) \sum_{j=1}^{n} \alpha(j)(\overline{s(j)} * f)(s(i)) =$$

$$\sum_{j=1}^{n} \alpha(j) \sum_{i=1}^{n} \lambda(i)(\overline{s(i)} * f)(s(j)) = \sum_{j=1}^{n} \alpha(j) = 1.$$

Consequently,

$$\sum_{i=1}^{n} \lambda(i)\overline{s(i)} * f = 1. \qquad \square$$

§2. Finite dimensional continuous representations.

In this section G will always denote a compact group in which the elements that have finite order are dense.

(5.15) DEFINITION. Let E be a vectorspace over K. A representation U of G is a homomorphism $x \to U_x$ of G into the semigroup of all operators on E.

That is, for each $x \in G$, U_x is an operator on E, and $U_{x+y} = U_x \circ U_y$ for all $x,y \in G$. If E is a topological vectorspace and the map $x \to U_x$ is continuous, then U is called a <u>continuous</u> representation. If E is finite dimensional, then U is called a <u>finite dimensional</u> representation. □

(5.16) THEOREM. Let $U : G \to L(E,E)$ be a finite dimensional, continuous representation. Then there is $e \in E$, $e \neq 0$, such that $U_x e = e$ for each $x \in G$.

<u>Proof</u>. Let E' be the dual space of E and let $U' : G \to L(E',E')$ be the dual representation [i.e. $(U'_x\Psi)(d) = \Psi(U_{-x}d)$ for each $x \in G$, $\Psi \in E'$, $d \in E$]. First assume that there is $\phi \in E'$ such that $E' = [\![\{U'_x\phi \mid x \in G\}]\!]$. Then choose $d \in E$ such that $\phi(d) = 1$. Define $f \in C(G)$ by the formula

$$f(x) = \phi(U_{-x}d) \qquad [x \in G].$$

If $s \in G$ and $U'_s\phi = \lambda(1)\, U'_{x(1)}\phi + \ldots + \lambda(n)\, U'_{x(n)}\phi$ $[\lambda(1),\ldots,\lambda(n) \in K$, $x(1),\ldots,x(n) \in G]$, then $\bar{s} * f = \lambda(1)\, \overline{x(1)} * f + \ldots + \lambda(n)\, \overline{x(n)} * f$. Hence, the space $[\![\{\bar{s} * f \mid s \in G\}]\!]$ is a finite dimensional shift invariant subspace of C(G). From (5.14) it follows that there are $\alpha(1),\ldots,\alpha(m) \in K$ together with $s(1),\ldots,s(m) \in G$ such that

$$1 = \alpha(1)\overline{s(1)} * f + \ldots + \alpha(m)\overline{s(m)} * f.$$

Define $e \in E$ by $e \underset{D}{=} \alpha(1)\, U_{s(1)}d + \ldots + \alpha(m)\, U_{s(m)}d$. Let $\Psi \in E'$ and let $\beta(1),\ldots,\beta(n) \in K$ and $x(1),\ldots,x(n) \in$ be such that

$$\Psi = \sum_{i=1}^{n} \lambda(i)\, U'_{x(i)}\, \Phi.$$

Then for each $s \in G$,

$$\Psi(U_{-s}e) = \sum_{i=1}^{n} \lambda(i)(U'_{x(i)}\phi)\ (U_{-s}e) = \sum_{i=1}^{n} \lambda(i) \sum_{j=1}^{m} (U'_{x(i)}\phi)(\alpha(j)U_{-s+s(j)}d))$$
$$\sum_{i=1}^{n}\lambda(i) \sum_{j=1}^{m} \alpha(j)\phi(U_{x(i)+s(j)+s}d) = \sum_{i=1}^{n} \lambda(i) \sum_{j=1}^{m} \alpha(j)(\overline{s(j)} * f)(x(i)+s)) =$$

$\sum_{i=1}^{n} \lambda(i)$. We deduce that $U_s e = e$ for each $s \in G$. From the fact that $\phi(e) = 1$ it follows that e is non-trivial. Now assume that

$$E' \neq [\![\{U'_x\Psi \mid x \in G\}]\!] \quad \text{for each } \Psi \in E'.$$

Choose $\chi \in E'$, $\chi \neq 0$ and let

$$D' = [\![\{U'_x\chi \mid x \in G\}]\!].$$

Put $D = \{e \mid e \in E,\ e(\Psi) = 0 \text{ for each } \Psi \in D'\}$. Then D is a non-trivial shift invariant linear subspace of E and E/D is non-trivial. By using an induction argument we see that there is a $d^{\sim} \in E/D$, $d^{\sim} \neq 0^{\sim}$, such that $(U_x d)^{\sim} = d^{\sim}$ for each $x \in G$. Choose a representative d of $d^{\sim}$. Now it can happen that $U_x d = d$ for each $x \in G$. Then we are done. Else, use again an induction argument to find

$$e \in [\![\{U_{x+y}d - U_x d \mid x \in G,\ y \in G\}]\!]$$

such that $e \neq 0$ and $U_x e = e$ for each $x \in G$. □

(5.17) REMARK. In (5.16) we may not drop the condition that E is finite dimensional.

<u>Counterexample</u>. Let $G = (C_p)^{\mathbb{N}}$. Put $H_0 = G$ and

$$H_n = \{x \mid x \in G,\ x_1 = \ldots = x_n = 0 \text{ for each } n \in \mathbb{N}\}.$$

Let $I = \{X \mid \text{there is } n \in \mathbb{N} \cup \{0\} \text{ such that } X \text{ is a coset of } H_n\}$. Then it is obvious that the formula $(U_x g)(h+H_n) \underset{D}{=} g(x+h+H_n)$ defines a continuous representation

$$U : G \to L(c_\infty(I), c_\infty(I)).$$

Let $\lambda \in K$ be such that $|\lambda| > 1$. Define $f \in c_\infty(I)$ by $f(x+H_n) = 0$ when $x \notin H_n$ and $f(H_n) = \lambda^{-n}$. Then it is easy to check that

$$f(x+H_n) = \lambda f(x+H_{n+1}) + \lambda f(x+a_{n+1}+H_n) + \ldots + \lambda f(x+(p-1)a_{n+1}+H_{n+1})$$

for each $n \in \mathbb{N} \cup \{0\}$ [where $a_1 = (1,0,\ldots), a_2 = (0,1,0,\ldots)$, etc.].

Let $E = \mathrm{clo}\,[\![\{U_s f \mid s \in G\}]\!]$ and let $U' : G \to L(E,E)$ be the representation induced by U.

Suppose that there is $h \in E$, $h \neq 0$, such that $U_x h = h$ for each $x \in G$. It is clear that h has the property that

$h(x+H_n) = \lambda h(x+H_{n+1}) + \lambda h(x+a_{n+1}+H_{n+1})+\ldots+ \lambda h(x+(p-1)a_{n+1}+H_{n+1})$.

It follows that $h(x+H_n) = 0$ for each $x \in G$, which is a contradiction. □

(5.18) REMARK. The example above also shows that there are infinite dimensional continuous representations without a finite dimensional, invariant subspace.[Also see D]. □

The following theorem shows that for the study of finite dimensional representations of a group G, it is important to study finite dimensional shift invariant subspaces of $(C(G))^n$ [for each $n \in \mathbb{N}$!].

(5.19) THEOREM. Let E be a finite dimensional vectorspace and let $U : G \to L(E,E)$ be a continuous representation. Then there is $n \in \mathbb{N}$ and a linear injection A from E into $(C(G))^n$ such that

$$A(U_x e) = (\bar{x} * (Ae)_1, \bar{x} * (Ae)_2, \ldots, \bar{x} * (Ae)_n) \quad [e \in E \text{ and } x \in G].$$

Proof. Denote $E_0 = [\![\{e \mid e \in E, U_x e = e \text{ all } x \in G\}]\!]$. Then from (5.16) we deduce that E_0 is non-trivial. Let $e_1,\ldots,e_n$ be a base of E_0 and choose $\phi_1,\ldots,\phi_n \in E'$ such that $\phi_i(e_j) = 0$ when $i \neq j$ and $\phi_i(e_i) = 1$ [E' is the dual space of E]. Define A by

$$(Ad)_i(x) \underset{D}{=} \phi_i(U_x d) \quad [d \in E,\ i \in \{i,\ldots,n\}].$$

Then it is easy to check that A is linear and that

$$A(U_x d) = (\bar{x} * (Ad)_1,\ldots,\bar{x} * (Ad)_n) \text{ for each } x \in G \text{ and } d \in E.$$

Now let $d \in E$ and $d \neq 0$. Choose $e \in [\![\{U_x d \mid x \in G\}]\!]$, $e \neq 0$, such that

$U_x e = e$ for each $x \in G$. There are $\lambda(1),\ldots,\lambda(n) \in K$ such that $e = \lambda(1)e_1 + \ldots + \lambda(n)e_n$. Then $Ae = (\lambda(1),\ldots,\lambda(n))$. Consequently, $Ae \neq 0$ and it follows that $A(U_x d) \neq 0$ for some $x \in G$. Hence, A is injective. $\square$

We need (5.20) in the proof of (5.23), but the lemma is interesting in its own right.

(5.20) LEMMA. Let E be a finite dimensional vectorspace and let $U : G \to L(E,E)$ be a continuous representation. Let D be a shift invariant subspace of E. Assume that $D \neq E$. Then there is an $e \in E$, $e \notin D$, such that $[\![e]\!] + D$ is shift invariant.

Proof. Choose $e \in E$, $e \notin D$ such that $U_x e - e \subset D$ for each $x \in G$ [use (5.16)]. Then e has the wanted properties. $\square$

For the next theorem we need some definitions.

(5.21) DEFINITION. Let E be a topological vectorspace and let $U : G \to L(E,E)$ be a continuous representation. Then E is called a function space if there is a linear continuous injection A from E into $C(G)$ such that $AU_x e = \bar{x} * Ae$ for each $e \in E$ and $x \in G$. $\square$

(5.22) DEFINITION. Let E be a topological vectorspace and let $U : G \to L(E,E)$ be a continuous representation. Then U is called cyclic if there is an $e \in E$ such that $\mathrm{clo}[\![\{U_x e \mid x \in G\}]\!] = E$. Such an e is a cyclic vector of E. $\square$

(5.23) THEOREM. Let E be a finite dimensional vectorspace and E' its dual space. Let $U : G \to L(E,E)$ be a continuous representation and $U' : G \to L(E',E')$ its dual representation. Then the following three statements are equivalent:

(i) $\dim[\![\{e \mid e \in E, U_x e = e \text{ all } x \in G\}]\!] = 1$;

(ii) E is a function space;

(iii) U' is cyclic.

Proof. (i) ⇒ (ii) follows from the proof of (5.19).

(ii) ⇒ (i) is obvious.

(iii) ⇒ (i). Suppose $\dim [\![\{e \mid e \in E, U_x e = e \text{ all } x \in G\}]\!] > 1$. Then choose $e_1, e_2 \in [\![\{e \mid e \in E, U_x e = e \text{ all } x \in G\}]\!]$ which are linearly independent. Let $\Psi \in E'$ be such that $\Psi(e_1) = 1$ and $\Psi(e_2) = 0$. Choose a cyclic vector ϕ of E'. Then there are $\lambda(1),\ldots,\lambda(n) \in K$ together with $x(1),\ldots,x(n) \in G$ such that $\Psi = \lambda(1)\, U'_{x(1)}\phi + \ldots + \lambda(n) U'_{x(n)}\phi$. Hence,
$1 = \Psi(e_1) = \lambda(1)\ (U'_{x(1)}\phi)(e_1) + \ldots + \lambda(n)\ (U'_{x(n)}\phi)(e_1) =$
$(\lambda(1)+\ldots+\lambda(n))\phi(e_1)$ and therefore $\lambda(1)+\ldots+\lambda(n) \neq 0$.
Further, $0 = \Psi(e_2) = (\lambda(1)+\ldots+\lambda(n))\phi(e_2)$ and we deduce that $\phi(e_2) = 0$. Consequently, by using that ϕ is a cyclic vector of E', it follows that $e_2 = 0$, which is a contradiction.

(i) ⇒ (iii). Let $e \in E$ be such that $e \neq 0$ and such that $U_x e = e$ for each $x \in G$. Denote $E'_0 = [\![\{\Psi - U'_x\Psi \mid \Psi \in E', x \in G\}]\!]$. Choose $\phi \in E'$ with $\phi(e) = 1$. We are going to prove that ϕ is a cyclic vector in E'. First note that $\phi \notin E'_0$. Hence, $\dim (E'/E'_0) \geq 1$. Suppose that $\dim (E'/E'_0) > 1$. Then choose $\chi \in E'$ such that $\chi^\sim$ and $\phi^\sim$ are linearly independent [where $\chi^\sim$ and $\phi^\sim$ are the residue classes in E'/E'_0 of χ and ϕ respectively]. Let $d \in E$ be such that $\psi(d) = 0$ for each $\psi \in E'_0$, such that $\phi(d) = 0$ and such that $\chi(d) = 1$. Then for each $\rho \in E'$ and each $x \in G$, $(U'_x\rho - \rho)(d) = 0$ and therefore $U_x d = d$ for each $x \in G$. However, d and e are linearly independent. Hence, we have a contradiction. Consequently, $\dim E'/E'_0 = 1$. It follows that it suffices to prove that

$D' \underset{D}{=} [\![\{U'_x\phi - \phi \mid x \in G\}]\!] = E'_0$. Suppose that $D' \neq E'_0$. D' is shift invariant and therefore we can find a shift invariant subspace F' of E'_0 with $D' \subset F'$ and $\dim E'_0/F' = 1$ [use (5.20)]. Then choose $\chi \in E'_0$ such that $[\![\phi,\chi]\!] + F' = E'$. It follows from (5.16) that $U'_x\chi - \chi \in F'$ for each $x \in G$. Let $d \in E$ be such that $\phi(d) = 0$, $\chi(d) = 1$ and $\Psi(d) = 0$ for each $\Psi \in F'$. Then for each $x \in G$ and each $\Psi \in F'$, $\phi(U_xd - d) = 0$, $\chi(U_xd - d) = 0$ and $\Psi(U_xd - d) = 0$. We infer that $U_xd = d$ for each $x \in G$. But d and e are linearly independent and we again have a contradiction. $\square$

(5.24) COROLLARY. Let E be a function space and let e be such that

$$[\![\{d \mid d \in E,\ U_xd = d \text{ all } x \subset G\}]\!] = [\![e]\!].$$

Let $\phi \in E'$. Then equivalent are:

(i) ϕ is a cyclic vector;

(ii) $\phi(e) \neq 0$.

Proof. Follows from the proof of the implication (i) $\Rightarrow$ (iii) in (5.23). $\square$

CHAPTER VI

FINITE DIMENSIONAL REPRESENTATIONS OF C_2^I

In this chapter we always assume that the characteristic of K is 2. G always denotes a compact group isomorphic to C_2^I for some index set I. We are going to study the finite-dimensional representations of C_2^I. Main results are (6.9), (6.28) and (6.30). The analogues of (6.1), (6.2) and (6.11) are also true in the more general case [i.e. if the characteristic of K is a prime number p and if G is a compact, p-primary group for which the elements that have finite order are dense].

I am not able yet to formulate and to prove analogues of the other theorems in the more general case.

(6.1) LEMMA. Let H be a finite subgroup of G. Let $f \in C(G)$ be such that $\bar{H} * f \neq 0$ [i.e. $\sum_{x\in H} \bar{x} * f \neq 0$]. Then the functions $(\bar{x} * f)_{x\in H}$ are linearly independent.

Proof. Let $(\alpha_x)_{x\in H}$ be a family of coefficients such that $\sum_{x\in H} \alpha_x.\bar{x} * f = 0$. Choose $y \in G$ such that $(\bar{H} * f)(y) \neq 0$. Define $\mu \in M(H)$ by $\mu(\{x\}) = f(x+y)$. Then $\mu(H) = \sum_{x\in H} f(x+y) = (\bar{H} * f)(y) \neq 0$. Hence, μ is invertible [see (5.4)]. Denote the inverse of μ by ν. Then $\sum_{x\in H} \alpha_x.\nu * \bar{x} * f = 0$. Now let $s \in H$. Then

$$\sum_{x\in H} \alpha_x(\nu * \bar{x} * f)(y + s) = \alpha_s.$$

Consequently, $\alpha_s = 0$ for each $s \in H$ and the proof is done. □

(6.2) COROLLARY. Let $f \in C(G)$ be such that the linear space $[\{\bar{s} * f \mid s \in G\}]$ is finite dimensional. Then there is $N \in \mathbb{N}$ such that $\bar{L} * f = 0$ for every finite subgroup L of G with $\#L > 2^N$. □

(6.3) DEFINITION. $f \in C(G)$ is called a <u>homomorphism</u> if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x,y \in G. \quad \square$$

(6.4) THEOREM. Put $H_i = \{x \mid x \in G, x_i = 0\}$ [each $i \in I$]. $h_i \in G$ is the element for which the i^{th} coordinate is non-trivial and all the other coefficients are trivial. Let $(\lambda_i)_{i\in I} \in c_\infty(I)$. Then $f \in C(G)$ defined as

$$f = \sum_{i\in I} \lambda_i \zeta(h_i+H_i)$$

is a homomorphism. Conversely, let $g \in C(G)$ be a homomorphism. Then there is $(\beta_i)_{i\in I} \in c_\infty(I)$ such that

$$g = \sum_{i\in I} \beta_i \zeta(h_i+H_i).$$

<u>Proof</u>. From the facts that $[G:H_i] = 2$ for each $i \in I$ and that the characteristic of K is 2, it follows immediately that $\zeta(h_i+H_i)$ is a homomorphism for each $i \in I$ and we infer that f is a homomorphism. To prove the converse, first remark that $g(0) = 0$ [$g(0) = g(0) + g(0)$!]. Define for each $i \in I$, β_i by $\beta_i = g(h_i)$. From the continuity of g it follows that $(\beta_i)_{i\in I} \in c_\infty(I)$. Then from the first part of the proof it follows that the function $\sum_{i\in I} \beta_i \zeta(h_i+H_i)$ is a homomorphism. The subgroup generated by the set $\{h_i \mid i \in I\}$ is dense in G. Hence, to prove that

$$g = \sum_{i\in I} \beta_i \zeta(h_i+H_i)$$

it suffices to prove that

$$g(h_j) = (\sum_{i\in I} \beta_i \zeta(h_i+H_i))(h_j) \text{ for each } j \in I.$$

But this is almost trivial. □

(6.5) THEOREM. Let $N \in \mathbb{N}$ and let $f \in C(G)$ be such that $\bar{L} * f = 0$ as soon as L is a finite subgroup of G with $\#L > 2^N$. Let H be a subgroup of G with $\#H = 2^{N-1}$. Define $g \in [\![\{\bar{s} * f \mid s \in G\}]\!]$ by $g = \bar{H} * f + (\bar{H}*f)(0).1$. Then g is a homomorphism.

Proof. Choose $x, y \in G$. First assume that $x, y \in H$. Then
$g(x+y) + g(x) + g(y) = (\bar{H}*f)(x+y) + (\bar{H}*f)(x) + (\bar{H}*f)(y) + (\bar{H}*f)(0) =$
$(\bar{H}*f)(0) + (\bar{H}*f)(0) + (\bar{H}*f)(0) + (\bar{H}*f)(0) = 0$. Secondly, assume that $x \notin H$ and $y \in [\![x]\!] + H$. Let $y = r.x + h$ where $h \in H$ and $r \in \{0,1\}$. Then
$g(x+y) + g(x) + g(y) = (\bar{H}*f)(x+y) + (\bar{H}*f)(x) + (\bar{H}*f)(y) + (\bar{H}*f)(0) =$
$(\bar{H}*f)(x+rx) + (\bar{H}*f)(x) + (\bar{H}*f)(rx) + (\bar{H}*f)(0) = 0$.
Finally, assume that $x \notin H$ and $y \notin [\![x]\!] + H$. Then $\#[\![x,y]\!] + H = 2^{N+1}$.
Consequently, $g(x+y) + g(x) + g(y) = ((\overline{[\![x,y]\!] + H}) * f)(0) = 0$. □

(6.6) COROLLARY. Let $f \in C(G)$ be such that the linear space $[\![\{\bar{s} * f \mid s \in G\}]\!]$ is finite dimensional. Assume that $\dim[\![\{\bar{s} * f \mid s \in G\}]\!] \geq 2$. Then there is a non-zero homomorphism in $[\![\{\bar{s} * f \mid s \in G\}]\!]$.

Proof. Use (6.2) and the fact that $\dim [\![\{\bar{s} * f \mid s \in G\}]\!] \geq 2$ to deduce that there is a minimal $N \in \mathbb{N}$ such that $\bar{L} * f = 0$ for each finite subgroup L of G with $\#L > 2^N$. Choose a subgroup H of G with $\#H = 2^N$ and with $\bar{H} * f \neq 0$. Let H_0 be a subgroup of H with $\#H_0 = 2^{N-1}$. Then $\bar{H}_0 * f + (\bar{H}_0*f)(0).1$ is a non-trivial element of $[\![\{\bar{s} * f \mid s \in G\}]\!]$. Further, by using (6.5) we see that $\bar{H}_0 * f + (\bar{H}_0*f)(0).1$ is a homomorphism. □

(6.7) LEMMA. Let $g_1,\ldots,g_n \in C(G)$ be homomorphisms. Define $g \in C(G)$ as

$g = \prod_{D\, i=1}^{n} g_i$. Then $\dim[\![\{\bar{s} * g \mid s \in G\}]\!] \leq 2^n$.

Proof. Let $s \in G$. Then for each $j \in \{1,\ldots,n\}$, $\bar{s} * g_j = g_j + g_j(s).1$. Hence, $\bar{s} * \prod_{i=1}^{n} g_i = \prod_{i=1}^{n} (g_i + g_i(s).1)$. It follows that $\bar{s} * g \in [\![\{ \prod_{j \in J} g_j | J \subset \{1,\ldots,n\}\}]\!]$ [where $\prod_{\emptyset} g_j = 1$] . $\#\{ \prod_{j\in J} g_j \mid J \subset \{1,\ldots,n\}\} \leq 2^n$. Now the lemma follows at once. □

<u>From now on</u> A <u>denotes the subalgebra of</u> C(G) <u>generated by the homomorphisms</u>.

(6.8) LEMMA. Let $f \in A$. Then $\dim[\![\{\bar{s} * f \mid s \in G\}]\!] < \infty$.

Proof. From (6.7) we infer that it suffices to prove that

$$[\![\{\bar{s} * (h_1+\ldots h_n) \mid s \in G\}]\!]$$

is finite dimensional when $h_1,\ldots,h_n \in C(G)$ are such that $[\![\{\bar{s} * h_i \mid s \in G\}]\!]$ is finite dimensional for each $i \in \{1,\ldots,n\}$. But this is almost trivial.

□

In (6.9) - (6.25) we prove the converse of (6.8).

<u>In</u> (6.9) - (6.25) $f \in C(G)$ <u>will be fixed such that</u> $\dim[\![\{\bar{s} * f \mid s \in G\}]\!] < \infty$. $N \in \mathbb{N}$ <u>is such that</u> $\bar{L} * f = 0$ <u>as soon as</u> L <u>is a finite subgroup of</u> G <u>with</u> $\#L \geq 2^{N+1}$ [see (6.2)].

(6.9) THEOREM. $f \in A$.

Proof. In the case that $N = 1$, f is a constant function and therefore in A [see (5.11)].

Now assume that $N > 1$. In (6.10) - (6.25) we prove that there is an element $k \in A$ such that $\bar{L} * (f+k) = 0$ as soon as L is a finite subgroup of G with $\#L \geq 2^N$. $[\![\{\bar{s} * (f+k) \mid s \in G\}]\!]$ is finite dimensional and according to this fact it follows, by using an induction argument, that $f + k \in A$. Hence, $f \in A$. □

First let us set up some more terminology.

(6.10) DEFINITION. Let H be a finite subgroup of G and let $g_1, g_2 \in C(G)$. Then g_2 is called a <u>shifting of</u> g_1 <u>over</u> H if there is a function α on G that is zero outside H such that $g_2 = \alpha * g_1$. g_2 is called a <u>one-to-one shifting of</u> g_1 <u>over</u> H if g_2 is a shifting of g_1 over H and g_1 is a shifting of g_2 over H. □

(6.11) LEMMA. Let H be a finite subgroup of G and let $g_1 \in C(G)$ be such that $(\bar{H} * g_1)(0) \neq 0$. Let α be a function on H. Then there is $g_2 \in C(G)$ which is a shifting of g_1 over H, such that $g_2(x) = \alpha(x)$ for each $x \in H$. When $\sum_{x \in H} \alpha(x) \neq 0$, for g_2 we can choose a one-to-one shifting of g_1 over H.

<u>Proof</u>. Let $\beta \in C(H)$ be defined as $\beta(x) = g_1(x)$ for each $x \in H$. Then $(\bar{H} * \beta)(0) = (\bar{H} * g_1)(0) \neq 0$. Therefore, $\bar{H} * \beta \neq 0$ and according to (6.1) the functions $(\bar{x} * \beta)_{x \in H}$ are linearly independent. Hence, the set $\{\bar{x} * \beta \mid x \in H\}$ is a base of C(H). Consequently, there are coefficients $(\lambda_x)_{x \in H}$ such that $\alpha = \sum_{x \in H} \lambda_x (\bar{x} * \beta)$. Define $\rho \in C(H)$ by $\rho(x) = \lambda_x$ for each $x \in H$. Then we infer that $g_2 \underset{D}{=} \rho * g_1$ is a shifting of g_1 over H. The second part of the lemma follows immediately from the first part. □

(6.12) NOTATION. Let L be a finite subgroup of G with $\#L \leq 2^N$ and let $l \in C(G)$. Then $L^* * l \underset{D}{=} 0$ if $\#L < 2^N$ and $L^* * l \underset{D}{=} \bar{L} * l$ if $\#L = 2^N$. [It should be kept in mind that, throughout (6.9) - (6.25), N is a fixed positive number.] □

(6.13) CONSTRUCTION OF k.

Let $H \subset G$ be a finite subgroup of G such that the set $\{\bar{x} * f \mid x \in H\}$ is complete in $[\![\{\bar{s} * f \mid s \in G\}]\!]$. Put $H = \{x_0,\dots,x_n\}$ where $x_0 = 0$. Consider the set $P = \{(p(1),\dots,p(N)) \mid p(i) \in \{1,\dots,n\},\ p(1) < p(2)<\dots<p(N)\} \cup \{0,\dots,0)\}$. A total ordering $<$ on P is defined by

$$(p(1),\dots,p(N)) < (q(1),\dots,q(N))$$

if the last non-zero entry of the sequence $(q(1)- p(1),\dots,q(N)-p(N))$ is positive. For each $p \in P$ and $r \in \{1,\dots,N\}$, $p = (p(1),\dots,p(N))$, put $H_p = [\![x_{p(1)},\dots,x_{p(N)}]\!]$ and $H_{p(r)} = [\![x_{p(1)},\dots,x_{p(r-1)},x_{p(r+1)},\dots,x_{p(N)}]\!]$.

The set $\{f_p \mid p \in P\}$ is defined by induction relative to p:

(i) Let $p = (0,\dots,0)$. Then $f_p \underset{D}{=} f$.

(ii) Let $p \in P$ have a direct predecessor $p - 1$. Then

(a) let $(H_p^* * f_{p-1})(0) = 0$. Then define f_p by $f_p = f_{p-1}$.

(b) let $(H_p^* * f_{p-1}) \neq 0$. Let $p = (p(1),\dots,p(N))$.

From (6.11) we deduce that there is a function g_{p-1}, which is a one-to-one shifting of f_{p-1} over H_p, such that $g_{p-1}(x) = 0$ when $x \in H_p$, $x \neq x_{p(1)}+\dots+x_{p(N)}$ and $g_{p-1}(x_{p(1)}+\dots+x_{p(N)}) = 1$. For each $r \in \{1,\dots,N\}$ define $a_{p(r)}$ by

$$a_{p(r)} \underset{D}{=} \bar{H}_{p(r)} * g_{p-1}.$$

Then f_p is defined as

$$f_p = g_{p-1} + \prod_{r=1}^{N} a_{p(r)}.$$

In (6.14) - (6.17) we show that for each $j \in P$ there is a function $k_j \in A$, together with a shifting h_j of f_j over H such that $f + k_j = h_j$. Denote the largest element of P by i_0. Then define k by

$$k_D = k_{i_o}. \qquad \square$$

<u>In</u> (6.14) - (6.25) <u>everything will be as in</u> (6.13), i.e. <u>the definitions of</u> $P, H, H_p, H_{p(r)}, f_p, g_p, a_{p(r)}, k_p$ <u>and</u> h_p <u>are as in</u> (6.13).

(6.14) LEMMA. Let L be a finite subgroup of G with $\#L \geq 2^{N+1}$. Then $\bar{L} * f_p = 0$ for each $p \in P$.

<u>Proof</u>. By induction relative to $p \in P$. The proof is the case where $p = (0,\ldots,0)$ is trivial $[f = f_{(0,\ldots,0)}!]$. Now assume that $p \in P$ has a direct predecessor $p - 1$. Let

$$(H^*_{p-1} * f_{p-1})(0) = 0.$$

Then the assertion follows by using an induction argument. $[f_p = f_{p-1}!]$. Therefore, assume that $(H^*_{p-1} * f_{p-1})(0) \neq 0$. g_{p-1} is a shifting of f_{p-1} over H_p. Hence, by using the induction argument, it follows that $\bar{L} * g_{p-1} = 0$. Then from (6.4) we infer that $a_{p(r)}$ is a homomorphism for each $r \in \{1,\ldots,N\}$. Now combine (6.1) and (6.7) and we see that

$$\bar{L} * \prod_{r=1}^{N} a_{p(r)} = 0.$$

Consequently,

$$\bar{L} * f_p = \bar{L} * g_{p-1} + \bar{L} * \prod_{r=1}^{N} a_{p(r)} = 0. \qquad \square$$

From the proof of (6.14) we obtain

(6.15) LEMMA. Let $p \in P$, $p \neq (0,\ldots,0)$ be such that $(H^*_p * f_{p-1})(0) \neq 0$. Then $a_{p(r)}$ is a homomorphism for each $r \in \{1,\ldots,N\}$. $\square$

(6.16) follows easily from (6.15) and from the definitions of $a_{p(r)}$.

(6.16) LEMMA. Let $p \in P$, $p = (p(1),\dots,p(N))$, $p \neq (0,\dots,0)$, be such that $(H_p^* * f_{p-1})(0) \neq 0$. Let $r \in \{1,\dots,N\}$. Then

$$a_{p(r)}(x) = 0 \quad \text{all } x \in H_p \setminus x_{p(r)} + H_{p(r)}$$

and

$$a_{p(r)}(x) = 1 \quad \text{all } x \in x_{p(r)} + H_{p(r)}.$$

Hence

$$\left(\prod_{r=1}^{N} a_{p(r)}\right)(x) = 0 \text{ for all } x \in H_p,\ x \neq x_{p(1)}+\dots+x_{p(N)}$$

and

$$\left(\prod_{r=1}^{N} a_{p(r)}\right)(x_{p(1)}+\dots+x_{p(r)}) = 1. \qquad \square$$

(6.17) LEMMA. For each $p \in P$ there is a function $k_p \in A$ together with a shifting h_p of f_p over H such that $f + k_p = h_p$.

Proof. Again by induction relative to $p \in P$. The case where $p = (0,\dots,0)$ is trivial. Now let $p \in P$ have a direct predecessor $p - 1$. In the case that $(H_p^* * f_{p-1})(0) = 0$, define k_p and h_p as k_{p-1} and h_{p-1} respectively.

In the case that $(H_p^* * f_{p-1})(0) \neq 0$, let α be a function on H_p such that $f_{p-1} = \alpha * g_{p-1}$. Let β be a function on H such that $\beta * f_{p-1} = k_{p-1} + f$. Define h_p as $h_p \underset{D}{=} \beta * \alpha * f_p$ and k_p as

$$k_p \underset{D}{=} k_{p-1} + \beta * \alpha * \prod_{r=1}^{N} a_{p(r)}.$$

Then h_p is a shifting of f_p over H, $k_p \in A$ and $h_p = k_p + f$. $\square$

This completes the construction of the function k $[\underset{D}{=} k_{i_0}$ where i_0 is the greatest element of P!$]$.

Now we are going to show that $\bar{L} * (f + k) = 0$ for each finite subgroup L of G with $\#L \geq 2^N$ [see (6.18) - (6.25)].

(6.18) LEMMA. Let $i, j \in P$ such that $j \leq i$. Then

$$(H_j^* * f_i)(0) = 0.$$

Proof. By induction relative to $i \in P$. The case where $i = (0,\dots,0)$ is almost trivial [$\#H_{(0,\dots,0)} = 1$ and therefore $H^*_{(0,\dots,0)} * f_i = 0$!]. Therefore, assume that i has a direct predecessor $i - 1$.

Case 1. $(H_i^* * f_{i-1})(0) = 0$. Then $f_i = f_{i-1}$ and it follows, by using the induction hypothesis, that $(H_j^* * f_i)(0) = 0$ for each $j \leq i$.

Case 2. $(H_i^* * f_{i-1})(0) \neq 0$.

By using (6.16) and the definition of g_{i-1} it follows immediately that $(H_i^* * f_i)(0) = (H_i^* * g_{i-1})(0) + (H_i^* * \prod_{r=1}^{N} a_{i(r)})(0) = 0$.

In the case where $j < i$ and $\#H_j < 2^N$, it is also almost trivial that $(H_j^* * f_i)(0) = 0$.

Hence, what is left is the proof of the fact that $(H_j^* * f_i)(0) = 0$ in the case where $\#H_j = 2^N$ and $j < i$.

Let α be a function on H_i such that $g_{i-1} = \alpha * f_{i-1}$. We prove that $(H_j^* * g_{i-1})(0) = 0$ [(2.a)] and that $(H_j^* * \prod_{r=1}^{N} a_{i(r)})(0) = 0$ [(2.b)].

Then from the definition of f_i it follows that $(H_j^* * f_i)(0) = 0$.

(2.a) $(H_j^* * g_{i-1})(0) = (H_j^* * \alpha * f_{i-1})(0) = (\bar{H}_j * \alpha * f_{i-1})(0) =$

$\sum_{x \in H_i} \alpha(x)(\bar{H}_j * f_{i-1})(x)$. It follows that it is sufficient to prove that $(\bar{H}_j * f_{i-1})(x) = 0$ for each $x \in H_i$.

Let $x \in H_j$. Then from the induction hypothesis we infer

$$(\bar{H}_j * f_{i-1})(x) = (\bar{H}_j * f_{i-1})(0) = (H_j^* * f_{i-1})(0) = 0.$$

Now let $x \notin H_j$. From (6.14) it follows that $(\overline{[x] + H_j} * f_{i-1})(0) = 0$.

Hence,

$$(\bar{H}_j * f_{i-1})(x) = (\bar{H}_j * f_{i-1})(0) = 0.$$

(2.b) Put $\imath = (q(1),\dots,q(N))$ and $\jmath = (p(1),\dots,p(N))$. Let $t \in \{1,\dots,N\}$ be such that $p(r) = q(r)$ when $r > t$ and $p(t) < q(t)$. I prove that $a_{q(t)}(x) = 0$ for each $x \in H_\jmath$.

As $a_{q(t)}$ is a homomorphism, it suffices to prove that $a_{q(t)}(x_{p(r)}) = 0$ for each $r \in \{1,\dots,N\}$.

First let $r > t$. Then $x_{p(r)} \in H_{\imath(t)}$ and we can use (6.16).

Now let $r \le t$. Then $x_{p(r)} \in H_{\imath(t)}$ or there is $\jmath' \in P$, $\jmath' < \imath$, such that $H_{\jmath'} = [\![x_{q(1)},\dots,x_{q(t-1)},x_{p(r)},x_{q(t+1)},\dots,x_{q(N)}]\!]$. In the first case use (6.16). In the second case, it follows from the conduction hypothesis that

$a_{q(t)}(x_{p(r)}) = (\overline{H_{\imath(t)}}*g_{\imath-1})(x_{p(r)}) = (\overline{H_{\imath(t)}}*g_{\imath-1})(x_{p(r)})+(H_{\imath(t)}*g_{\imath-1})(0) = H_{\jmath'} * g_{\imath-1})(0) = 0.$ □

(6.19) LEMMA. Let $m \in \mathbb{N}$ and let $\mu \in M(C_2^m)$. $L_1,\dots,L_m$ are subgroups of C_2^m such that $[C_2^m:L_\imath] = 2$ for each $\imath \in \{1,\dots,m\}$ and such that $\bigcap_{\imath=1}^m L_\imath = \{0\}$. Suppose that $\mu(C_2^m) = 0$ and $\mu(L_\imath) = 0$ for each $\imath \in \{1,\dots,m\}$. Then if L is a subgroup of C_2^m with $[C_2^m:L] = 2$, then $\mu(L) = 0$.

Proof. Let $l \in C_2^m \setminus L$ and $l_\imath \in C_2^m \setminus L_\imath$ [$\imath \in \{1,\dots,m\}$]. To prove that $\mu(L) = 0$ it suffices to prove that $\mu(l+L) = 0$ [$\mu(C_2^m) = 0$!]. For this it suffices to prove that $\zeta(l+L)$ is a linear combination of the functions $\zeta(l_1+L_1),\dots,\zeta(l_m+L_m)$. Let V be the linear space of homomorphism $C_2^m \to K$. It is clear that $\zeta(l+L),\zeta(l_1+L_1),\dots,\zeta(l_m+L_m) \in V$. It follows that we are done when we can prove that $\zeta(l_1+L_1),\dots,\zeta(l_m+L_m)$ is a base of V. It follows from (6.4) that $\dim V = m$. Therefore, we are done when we can prove that the sequence $\zeta(l_1+L_1),\dots,\zeta(l_m+L_m)$ is linearly independent. But this follows immediately from the fact that $\bigcap_{\imath=1}^m L_\imath = \{0\}$. □

(6.20) LEMMA. Let $\imath \in P$, $\imath = (q(1),\dots,q(N))$, $\imath \ne (0,\dots,0)$ be such that $(H_\imath^* * f_{\imath-1})(0) \ne 0$. Denote $H_{\imath(0)} \underset{D}{=} [\![x_{q(1)}+x_{q(2)},x_{q(2)},\dots,x_{q(N)}]\!]$.

Then $\overline{H_{i(r)}} * f_i = 0$ for each $r \in \{0,\dots,N\}$.

Proof. First we prove that $\bar{H}_{i(r)} * f_i = 0$ for each $r \in \{1,\dots,N\}$ [see (a)]. Then by using (6.19) and (a) we prove that $\bar{H}_{i(0)} * f_i = 0$ [(b)].

(a) $\bar{H}_{i(r)} * f_i = \bar{H}_{i(r)} * g_{i-1} + \bar{H}_{i(r)} * \prod_{j=1}^{N} a_{i(j)} = a_{i(r)} + \bar{H}_{i(r)} * \prod_{j=1}^{N} a_{i(j)}$.

Hence, we have to prove that

$$\bar{H}_{i(r)} * \prod_{j=1}^{N} a_{i(j)} = a_{i(r)}.$$

Let $s \in H_{i(r)}$ and $j \in \{1,\dots,N\}$. Then

$$\bar{s} * (a_{i(j)}) = a_{i(j)}(s).1 + a_{i(j)}.$$

Consequently,

$$\bar{H}_{i(r)} * \prod_{j=1}^{N} a_{i(j)} = \Sigma\{(\prod_{j\notin J} a_{i(j)}(s))(\prod_{j\in J} a_{i(j)}) \mid J \subset \{1,\dots,N\},\ s \in H_{i(r)}\} \quad (*)$$

[where $\prod_{j\in J D} = 1$ when $J = \emptyset$]. Let $J \in \{1,\dots,N\}$ and let us calculate

$$\Sigma\{\prod_{j\notin J} a_{i(j)}(s) \prod_{j\in J} a_{i(j)} \mid s \in H_{i(r)}\}.$$

(a.i) $J = \{1,\dots,N\}$. We have assumed that $N > 1$ [see (6.9)]. From $(H_i^* * f_i)(0) \neq 0$ it follows that $\# H_i = 2^N$. We infer that $\# H_{i(r)} = 2^{N-1} \geq 2$.

Hence,

$$\Sigma\{\prod_{j\notin J} a_{i(j)}(s) \prod_{j\in J} a_{i(j)} \mid s \in H_{i(r)}\} =$$

$$\Sigma\{\prod_{j\in J} a_{i(j)} \mid s \in H_{i(r)}\} = 0.$$

(a.ii) $J = \emptyset$. Then

$$\Sigma\{\prod_{j\notin J} a_{i(j)}(s) \prod_{j\in J} a_{i(j)} \mid s \in H_{i(r)}\} =$$

$$\Sigma\{\prod_{j=1}^{N} a_{i(j)}(s)\ .1 \mid s \in H_{i(r)}\} = 0 \text{ [use (6.16)]}.$$

(a.iii) $2 \leq \#J < N$. Let $s \in H_{i(r)}$. Then

$$\prod_{j \notin J} a_{i(j)}(s) = 1 \quad \text{if} \quad s \in \sum_{j \notin J} x_{q(j)} + [\![\{x_{q(j)} \mid j \in J\}]\!]$$

and

$$\prod_{j \notin J} a_{i(j)}(s) = 0 \quad \text{if } s \notin \sum_{j \notin J} x_{q(j)} + [\![\{x_{q(j)} \mid j \in J\}]\!].$$

The group $[\![\{x_{q(j)} \mid j \in J\}]\!]$ has an even number of elements and we deduce that

$$\Sigma\{(\prod_{j \notin J} a_{i(j)}(s))(\prod_{j \in J} a_{i(j)}) \mid s \in H_{i(r)}\} = 0.$$

(a.iv) $\#J = 1$ and $J \neq \{r\}$. Let $J = \{t\}$.

The function $a_{i(t)}$ vanishes identically on $H_{i(r)}$ and therefore the function $\prod_{j \in J} a_{i(j)}$ vanishes identically on $H_{i(r)}$. It follows that

$$\Sigma\{(\prod_{j \neq t} a_{i(j)}(s)) a_{i(t)} \mid s \in H_{i(r)}\} = 0.$$

(a.v) $J = \{r\}$. Then $\prod_{j \neq r} a_{i(j)}(s) = 0$ for each $s \in H_{i(r)}$, $s \neq x_{q(1)} + \ldots + x_{q(r-1)} + x_{q(r+1)} + \ldots + x_{q(N)}$ and

$(\prod_{j \neq r} a_{i(j)})(x_{q(1)} + \ldots + x_{q(r-1)} + x_{q(r+1)} + \ldots + x_{q(N)}) = 1$. Consequently,

$$\Sigma\{(\prod_{j \neq r} a_{i(j)}(s)) a_{i(r)} \mid s \in H_{i(r)}\} = a_{i(r)}.$$

Combining (a.i), (a.ii), (a.iii), (a.iv) and (a.v) we deduce that

$$\Sigma\{(\prod_{j \notin J} a_{i(j)}(s)) \prod_{j \in J} a_{i(j)} \mid s \in H_{i(r)},\ J \subset \{1,\ldots,N\}\} = a_{i(r)}.$$

(b) Let $z \in G$. Define $\mu \in M(H_i)$ by the formula

$$\mu(\{s\}) \underset{D}{=} f_i(s + z) \qquad [s \in H_i].$$

Then for each $r \in \{1,\ldots,N\}$, $\mu(H_{i(r)}) = (\bar{H}_{i(r)} * f_i)(z) = 0$ [use (a)].

Further, $\bigcap_{r=1}^{N} H_{i(r)} = \{0\}$. Now use (6.19) to deduce that $\mu(H_{i(0)}) = 0$.

Hence,

$$(\bar{H}_{i(0)} * f_i)(z) = \mu(H_{i(0)}) = 0.$$

We infer that $\bar{H}_{i(0)} * f_i = 0$. □

(6.21) Lemma. Let $i \in P$. Denote

$$D_i = [\{ \prod_{j \in J} a_{t(j)} \mid J \subset \{1,\dots,N\},\ \#J \leq N-2;\ t \in P,\ t \leq i\}].$$

Then D_i is shift invariant. Further, let $l \in D_i$. Then $\bar{L} * l = 0$ for each subgroup L of G with $\#L \geq 2^{N-1}$.

<u>Proof</u>. Let $s \in G$. Choose $J \subset \{1,\dots,N\}$ and $t \in P$ such that $\#J \leq N-2$ and $t \leq i$. Then $\bar{s} * \prod_{j \in J} a_{t(j)} = \prod_{j \in J} (a_{t(j)}(s).1 + a_{t(j)}) \in D_i$. The second part of the lemma follows from (6.1) together with (6.7). □

<u>In</u> (6.22) - (6.23), D_i <u>will be as defined in</u> (6.21).

(6.22) LEMMA. Let $i \in P$, $i = (q(1),\dots,q(N))$. Then

$$[\![\{\bar{s} * f_i \mid s \in G\}]\!] \subset [\![\{\bar{x} * f_i \mid x \in H\}]\!] + D_i.$$

<u>Proof</u>. Again, we may assume that $i > (0,\dots,0)$. [The case where $i = (0,\dots,0)$ is trivial!] Denote the direct predecessor of i by $i-1$. In the case that $(H_i^* * f_{i-1})(0) = 0$, the lemma follows by using an induction argument. Therefore, suppose that $(H_i^* * f_{i-1})(0) \neq 0$. Then $f_i = g_{i-1} + \prod_{j=1}^{N} a_{i(j)}$ (i). g_{i-1} is a one-to-one shifting of f_{i-1} over H_i. Further $H_i \subset H$. We deduce that $[\![\{\bar{x} * f_{i-1} \mid x \in H\}]\!] = [\![\{\bar{x} * g_{i-1} \mid x \in H\}]\!]$. Hence, by using the induction hypothesis it follows that

$$[\![\{\bar{s} * g_{i-1} \mid s \in G\}]\!] = [\![\{\bar{s} * f_{i-1} \mid s \in G]\!] \subset [\![\{\bar{x} * g_{i-1} \mid x \in H\}]\!] + D_{i-1} \quad \text{(ii)}.$$

Now combine (i) and (ii) and the definition of D_i to deduce that

$$[\![\{\bar{s} * f_i \mid s \in G\}]\!] \subset [\![\{\bar{x} * g_{i-1} \mid x \subset H\}]\!] +$$

$$[\![\{\prod_{j\in J} a_{i(j)} \mid J \subset \{1,\ldots,N\},\ \#J \geq N-1\}]\!] + D_i \qquad \text{(iii)}.$$

Let $z \in G$. Then it follows from (iii) that there are coefficients $(\alpha(s))_{s\in H}$ and $(\beta(r))_{r\in\{0,\ldots,N\}}$ such that

$$\bar{z} * f_i + \sum_{s\in H} \alpha(s).(\bar{s} * g_{i-1}) + \sum_{r=0}^{N} \beta(r)1_r \in D_i \qquad \text{(iv)},$$

where $1_0 = {}_D \prod_{j=1}^{N} a_{i(j)}$ and $1_r = {}_D \prod_{j\neq r} a_{i(j)}$ $[r \in \{1,\ldots,N\}]$.

Now let $t \in \{0,\ldots,N\}$. $\#H_i = 2^N$ and therefore $\#H_{i(t)} = 2^{N-1}$ [where $H_{i(0)}$ is as in (6.20)]. Then from (iv), by using (6.20) and (6.21), it follows that

$$\sum_{s\in H} \alpha(s).(\bar{H}_{i(t)} * g_{i-1})(s) + \sum_{r=0}^{N} \beta(r)(\bar{H}_{i(t)} * 1_r)(0) = 0.$$

Case (a). $t \in \{1,\ldots,N\}$. From (6.16) it follows that $(\bar{H}_{i(t)} * 1_r)(0) = 0$ when $r \neq t$ and $(\bar{H}_{i(t)} * 1_t)(0) = 1$. Consequently,

$$\beta(t) = \sum_{s\in H} \alpha(s).(\bar{H}_{i(t)} * g_{i-1})(s) \quad \text{for each } t \in \{1,\ldots,N\} \qquad \text{(v)}.$$

Case (b). $t = 0$. Then from (6.16) it follows that

$(\bar{H}_{i(0)} * 1_0)(0) = 1$, $(\bar{H}_{i(1)} * 1_1)(0) = 1$, $(\bar{H}_{i(2)} * 1_2)(0) = 1$ and

$(\bar{H}_{i(r)} * 1_r)(0) = 0$ for each $r > 2$.

Hence,

$$\beta(0) + \beta(1) + \beta(2) = \sum_{s\in H} \alpha(s)(\bar{H}_{i(0)} * g_{i-1})(s) \qquad \text{(vi)}.$$

By combining (v) and (vi) we infer that

$$\beta(0) = \sum_{s\in H} \{\alpha(s)((\bar{H}_{i(0)} * g_{i-1})(s) + (\bar{H}_{i(1)} * g_{i-1})(s) + (\bar{H}_{i(2)} + g_{i-1})(s))\}.$$

Now for each $s \in H$, $(\bar{H}_{i(0)} * g_{i-1})(s) + (\bar{H}_{i(1)} * g_{i-1})(s) + (\bar{H}_{i(2)} * g_{i-1})(s)$

$= \Sigma\{g_{i-1}(x+s) \mid x \in H_{i(0)}\} + \Sigma\{g_{i-1}(x+s) \mid x \in H_{i(1)}\} + \Sigma\{g_{i-1}(x+s) \mid s \subset H_{i(2)}\}$

$= \Sigma\{g_{i-1}(x+s) \mid x \in H_i\}$. Now use the definition of g_{i-1} and (6.14) to infer that $\Sigma\{g_{i-1}(x+s) \mid x \in H_i\} = 1$.

Consequently, $\beta(0) = \sum_{s \in H} \alpha(s)$ (vii).

From (iv), by using (v) and (vii) it follows that

$$\bar{z} * f_i + \sum_{s \in H} \alpha(s)(\bar{s} * g_{i-1}) + \sum_{s \in H} \alpha(s) \,.\, 1_0 +$$

$$\Sigma\{\alpha(s)(\bar{H}_{i(r)} * g_{i-1})(s) \,.\, 1_r \mid s \in H,\ r \in \{1,\dots,N\}\} \in D_i$$

i.e. $\bar{z} * f_i + \Sigma\{\alpha(s)(\bar{s} * g_{i-1} + 1_0 + (\bar{H}_{i(r)} * g_{i-1})(s).1_r) \mid s \in H, r \in \{1,\dots,N\}\} \in D_i$

(viii).

Now for each $s \in H$,

$$\bar{s} * g_{i-1} + 1_0 + \Sigma\{(\bar{H}_{i(r)} * g_{i-1})(s) 1_r \mid r \in \{1,\dots,N\}\} + \bar{s} * f_i =$$

$$\bar{s} * g_{i-1} + \prod_{j=1}^{N} a_{i(j)} + \Sigma\{a_{i(r)}(s) \,.\, \prod_{j \neq r} a_{i(j)} \mid r \in \{1,\dots,N\}\} + \bar{s} * g_{i-1} +$$

$$\bar{s} * \prod_{j=1}^{N} a_{i(j)} = \Sigma\{(\prod_{j \notin J} a_{i(j)}(s))(\prod_{j \in J} a_{i(j)}) \mid J \subset \{1,\dots,N\},\ \#J < N-1\} \in D_i \quad \text{(ix)}.$$

Now combine (viii) and (ix) to deduce that

$$\bar{z} * f_i + \sum_{s \in S} \alpha(s) \bar{s} * f_i \in D_i$$

and the lemma follows. □

(6.23) LEMMA. Let $i \in P$ and let $1_1,\dots,1_m \in D_i$ be such that the set $\{1_1,\dots,1_m\}$ is a base of D_i. Choose $y_1,\dots,y_r \in H$ such that the set $\{f_i, \bar{y}_1 * f_i, \dots, \bar{y}_r * f_i, 1_1,\dots,1_m\}$ is a base of $[\![\{\bar{x} * f_i \mid x \in H\}]\!] + D_i$. Let $z \in G$ and let $\alpha(0),\dots,\alpha(r),\beta(1),\dots,\beta(m)$ be coefficients such that

$$\bar z * f_i = \sum_{j=0}^{r} \alpha(j)(\bar y_j * f_i) + \sum_{j=1}^{m} \beta(j) l_j \qquad [\text{where } \bar y_0 \underset{D}{=} 0].$$

Then $\sum_{j=0}^{r} \alpha(j) = 1$.

Proof. $f_i = \bar z * \bar z * f_i = \sum_{j=0}^{r} \alpha(j)(\bar y_j * z * f_i) + \sum_{j=1}^{m} \beta(j)(\bar z * l_j) =$

$$\sum_{j,t=0}^{r} \alpha(j)\alpha(t)(\bar y_t * \bar y_j * f_i) + \Sigma\{\alpha(j)\beta(t) l_t \mid j = 0,\dots,r;\ t = 1,\dots,m\} +$$

$$\sum_{j=1}^{m} \beta(j)(\bar z * l_j) \qquad (*).$$

The coefficient of $\bar y_t * \bar y_j * f_i$ in $(*)$ in the case where $t \neq j$ is $\alpha(t)\alpha(j) + \alpha(j)\alpha(t) = 0$. In the case where $t = j$, it is $(\alpha(j))^2$, D_i is shift invariant [see (6.21)]. It follows that

$$f_i = \Big(\sum_{j=0}^{r} (\alpha(j))^2\Big) . f_i + h, \text{ where } h \in D_i. \text{ But then } h = 0.$$

Consequently $\sum_{j=0}^{r} (\alpha(j))^2 = 1$ and therefore $\sum_{j=0}^{r} \alpha(j) = 1$. $\square$

In the proof of (6.24) we need a notation: Let $q \in \mathbb{N}$, let $s_1,\dots,s_q \in G$ and let $l \in C(G)$. Then $[\![s_1,\dots,s_q]\!]_q^* * l \underset{D}{=} 0$ when $\#[\![s_1,\dots,s_q]\!] < 2^q$ and $\underset{D}{=} \overline{[\![s_1,\dots,s_q]\!]} * l$ when $\#[\![s_1,\dots,s_q]\!] = 2^q$.

[Note that $[\![s_1,\dots,s_N]\!]_N^* * l = [\![s_1,\dots,s_N]\!]^* * l$.]

(6.24) LEMMA. Let L be a finite subgroup of G with $\#L \geq 2^N$. Then $\bar L * f_{i_0} = 0$ [recall that i_0 denotes the largest element of P.].

Proof. From (6.14) it follows that we may assume that $\#L = 2^N$. Choose $y_1,\dots,y_r$ and $l_1,\dots,l_m$ as in (6.23). Denote the set $\{0,\dots,r\}$ by R and the set $\{1,\dots,m\}$ by M. Let $z_1,\dots,z_N$ be such that $L = [\![z_1,\dots,z_N]\!]$. Put $L_0 \underset{D}{=} [\![0]\!]$ and $L_j \underset{D}{=} [\![z_1,\dots,z_j]\!]$ $[j \in \{1,\dots,N\}]$. The coefficients

$\alpha_0^j,\ldots,\alpha_r^j$ and $\beta_1^j,\ldots,\beta_m^j$ are such that

$$\bar{z}_j * f_{\imath_0} = \sum_{s\in R} \alpha_s^j(\bar{y}_s * f_{\imath_0}) + \sum_{s\in M} \beta_s^j 1_s.$$

For the moment let us assume that for each

$$j \in \{1,\ldots,N\},\ \sum_{z\in L_j} \bar{z} * f_{\imath_0} =$$

$$\Sigma\{\alpha_{s(1)}^1\ \alpha_{s(2)}^2,\ldots,\ \alpha_{s(j)}^j, [\![y_{s(1)},\ldots,y_{s(j)}]\!]_j^* * f_{\imath_0}\ |\ s(1),\ldots,s(j) \in R\} +$$
$$\Sigma\{\beta_{s(t)}^t \alpha_{s(1)}^1\cdot\ldots\cdot\alpha_{s(t-1)}^{t-1} [\![y_{s(1)},\ldots,y_{s(t-1)}, z_{t+1},\ldots,z_j]\!]_{j-1}^* * 1_{s(t)}\ |$$
$$t \in \{1,\ldots,j\};\ s(t) \in M;\ s(1),\ldots,s(t-1) \in R\}.$$

Then especially $\sum_{z\in L} \bar{z} * f_{\imath_0} = \Sigma\{\alpha_{s(1)}^1\cdot\ldots\cdot\alpha_{s(N)}^N\cdot[\![y_{s(1)},\ldots,y_{s(N)}]\!]_N^* * f_{\imath_0}\,|$ $s(1),\ldots,s(N) \in R\} + \Sigma\{\beta_{s(t)}^t\alpha_{s(1)}^1\cdot\ldots\cdot\alpha_{s(t-1)}^{t-1}[\![y_{s(1)},\ldots,y_{s(t-1)},\ldots,z_{t+1}$ $,\ldots,z_N]\!]_{N-1}^* * t \subset \{1,\ldots,N\};\ s(1),\ldots,s(t-1) \in R,\ s(t) \in M\}.$

By using (6.14) together with (6.18) it follows that $[\![y_{s(1)},\ldots,y_{s(N)}]\!]_N^* * f_{\imath_0} = 0$ for each choice of $s(1),\ldots,s(N) \in R$. By using (6.21) it follows that $[\![y_{s(1)},\ldots,y_{s(t-1)},z_{t+1},\ldots,z_N]\!]_{N-1}^* * 1_{s(t)} = 0$ for each choice of $s(1),\ldots,s(t-1) \in R$ and $s(t) \in M$. Consequently,

$$\bar{L} * f_{\imath_0} = \sum_{z\subset L} \bar{z} * f_{\imath_0} = 0.$$

What is left is the proof of the assumption. First let $j = 1$. Then

$$\sum_{z\in L_1} \bar{z} * f_{\imath_0} = f_{\imath_0} + \bar{z}_1 * f_{\imath_0} =$$

$f_{\imath_0} + \sum_{s(1)\in R} \alpha_{s(1)}^1(\bar{y}_{s(1)} * f_{\imath_0}) + \sum_{s(1)\subset M} \beta_{s(1)}^1 1_{s(1)}$. By using (6.23) the assertion follows.

Now let $j \in \{1,\ldots,N\}$. Then $\sum_{z\subset L_j} \bar{z} * f_{\imath_0} = \sum_{z\in L_{j-1}} \bar{z} * f_{\imath_0} +$ $\sum_{z\in z_j+L_{j-1}} \bar{z} * f_{\imath_0}$. By using an induction argument it follows that

$\sum_{z\in L_j} \bar{z} * f_{i_0} = \Sigma\alpha^1_{s(1)}\cdot\ldots\cdot\alpha^{j-1}_{s(j-1)}[\![y_{s(1)},\ldots,y_{s(j-1)}]\!]^*_{j-1} * f_{i_0} \mid$

$s(1),\ldots,s(j-1) \in R\} + \Sigma\{\beta^t_s\alpha^1_{s(1)}\ldots\alpha^{t-1}_{s(t-1)}[\![y_{s(1)},\ldots,y_{s(t-1)},z_{t+1},\ldots,z_{j-1}]\!]^*_{j-2} *$

$1_{s(t)} \mid t \in \{1,\ldots,j-1\};\ s(1),\ldots,s(t-1) \in R;\ s(t) \in M\} +$

$\Sigma\{\alpha^1_{s(1)}\ldots\alpha^j_{s(j)}[\![y_{s(1)},\ldots,y_{s(j-1)}]\!]^*_{j-1} * \bar{y}_{s(j)} * f_{i_0} \mid s(1),\ldots,s(j) \in R\} +$

$\Sigma\{\alpha^1_{s(1)}\ldots\alpha^{j-1}_{s(j-1)}\beta^j_{s(j)},[\![y_{s(1)},\ldots,y_{s(j-1})]\!]^*_{j-1} * 1_{s(j)} \mid s(1),\ldots,s(j-1) \in R;$

$s(j) \in M\} + \Sigma\{\beta^t_{s(t)}\alpha^1_{s(1)}\ldots\alpha^{t-1}_{s(t-1)}[\![y_{s(1)},\ldots,y_{s(t-1)},z_{t+1},\ldots,z_{j-1}]\!]^*_{j-2} *$

$z_j * 1_{s(t)} \mid s(1),\ldots,s(t-1) \in R,\ s(t) \in M;\ t \in \{1,\ldots,j-1\}\}.$

Now the assertion follows when we use (6.23) and the fact that

$[\![a_1,\ldots,a_{q-1}]\!]^*_{q-1} * 1 + [\![a_1,\ldots,a_{q-1}]\!]^*_{q-1} * \bar{a}_q * 1 = [\![a_1,\ldots,a_q]\!]^*_q * 1$ for

each $q \in \mathbb{N}$, $1 \in C(G)$ and $a_q, a_1,\ldots,a_{q-1} \in G$. $\square$

(6.25) COROLLARY. $\bar{L} * (f+k) = 0$ for each finite subgroup L of G with $\#L \geq 2^N$.

Proof. $f + k$ is a shifting of f_{i_0} [see (6.17)]. Now use (6.24). $\square$

Suppose that $a_1,\ldots,a_n$ is an orthonormal set of homomorphisms. Denote $B = \{\prod_{j\in J} a_j \mid J \subset \{1,\ldots,n\}\}$ [where, as usual, $\prod_{j\in\emptyset} a_j = 1$].

(6.26) THEOREM. B is an orthonormal base of $[\![\{\bar{s} * \sum_{j=1}^n a_j \mid s \in G\}]\!]$.

Proof. Put $V = [\![\{\bar{s} * \prod_{j=1}^n a_j \mid s \in G\}]\!]$. First we prove that B is an orthonormal set [see (a)]. Then we prove that $B \subset V$ [(b)]. By combining (a) and (b) the theorem follows.

(a) For each $i \in \{0,\ldots,n\}$, denote $B_i = \{\prod_{j\in J} a_j \mid J \subset \{0,\ldots,n\}\ \#J \leq i\}$ and $D_i = [\![B_i]\!]$.

We prove by induction with respect to i that B_i is an orthonormal base of D_i. The case where $i = 0$ is trivial. Now assume that i has direct predecessor $i - 1$. Each function in B_i has the property that its norm is at most 1. Hence, to prove the orthonormality of the set B_i it suffices to prove the following.

Let $\{\alpha_J \mid J \subset \{1,\ldots,N\},\ \#J \leq i\}$ be a set of coefficients with $\max\{\alpha_J \mid J \subset \{1,\ldots,n\},\ \#J \leq i\} = 1$. Then

$$||\Sigma\{\alpha_J(\prod_{j\in J} a_j) \mid J \subset \{1,\ldots,n\},\ \#J \leq i\}|| = 1.$$

Assume that this norm is smaller than 1.

If $|\alpha_J| < 1$ for each $J \subset \{1,\ldots,n\}$ with $\#J = i$, Then the induction hypothesis already leads to a contradiction. It follows, that we may as well assume that $|\alpha_{\{1,\ldots,i\}}| = 1$.

Choose $x \in G$. Then

$$||\Sigma\{\alpha_J(\bar{x} * \prod_{j\in J} a_j)|\ J \subset \{1,\ldots,n\},\ \#J \leq i\}|| < 1.$$

Consequently,

$$||\Sigma\{\alpha_J(\bar{0} + \bar{x}) * \prod_{j\in J} a_j \mid J \subset \{1,\ldots,n\},\ \#J \leq i\}|| < 1.$$

Now it is easy to see that

$$\Sigma\{\alpha_J(\bar{0} + \bar{x}) * \prod_{j\in J} a_j \mid J \subset \{1,\ldots,n\},\ \#J \leq i\}$$

is a linear combination of elements of B_{i-1}. Let us consider the coefficient $\beta(x)$ of $\prod_{j=2}^{n} a_j$ in this sum.

$$\beta(x) = \Sigma\{\alpha_J\ a_j(x) \mid \{2,\ldots,i\} \underset{\neq}{\subset} J,\ \{j,2,\ldots,i\} = J\}.$$

By using the induction hypothesis it follows that $|\beta(x)| < 1$.

The function $a_1,\ldots,a_n$ are continuous and G is compact. We deduce that

$$||\Sigma\{\alpha_J a_j \mid \{2,\ldots,i\} \underset{\neq}{\subset} J,\ \{j,2,\ldots,i\} = J\}|| < 1.$$

But this gives a contradiction with the facts that $|\alpha_{\{1,\dots,i\}}| = 1$ and that the set $a_1,\dots,a_n$ is orthonormal.

(b) Let $B \setminus \{\prod_{j=1}^{n} a_j\} = \{f_1,\dots,f_t\}$ [where $f_i \neq f_j$ when $i \neq j$]. For each $x \in G$, we have

$$\prod_{j=1}^{n} a_j + \bar{x} * \prod_{j=1}^{n} a_j = \Sigma\{(\prod_{j\notin J} a_j(x))(\prod_{j\in J} a_j) \mid J \subset \{1,\dots,n\},\ \#J \leq n-1\}.$$

Hence, we are done when we can prove that there are $x_1,\dots,x_t \in G^t$ such that

$$D_1(x_1,\dots,x_t) \underset{D}{=} \begin{vmatrix} f_1(x_1) & \dots & f_1(x_t) \\ \vdots & & \\ f_t(x_1) & \dots & f_t(x_t) \end{vmatrix} \neq 0.$$

But $D_1(x_1,\dots,x_t) = 0$ for each $(x_1,\dots,x_t) \in G^t$ would imply, by using the linear independence of the set $\{f_1,\dots,f_t\}$, that

$$D_2(x_2,\dots,x_t) \underset{D}{=} \begin{vmatrix} f_2(x_2) & \dots & f_2(x_2) \\ \vdots & & \\ f_t(x_2) & \dots & f_t(x_t) \end{vmatrix} = 0$$

for each $(x_2,\dots,x_t) \in G^{t-1}$. Proceeding in this way, we finally would get that $f_t(x_t) = 0$ for each $x_t \in G$, which is a contradiction. $\square$

(6.27) LEMMA. Let $a_1,\dots,a_n$, $b_0,\dots,b_m$ be a set of homomorphisms such that the set $\{a_1,\dots,a_n\}$ is orthonormal and such that $b_0 = 0$. Then there are homomorphisms $a_{n+1},\dots,a_k$ such that the set $\{a_1,\dots,a_k\}$ is orthonormal and such that

$$\prod_{j=1}^{m} b_j \in [\![\{\bar{x} * \prod_{j=1}^{k} a_j \mid x \in G\}]\!].$$

<u>Proof.</u> By induction to m. The case where $m = 0$ is trivial. Now let $m > 0$. There are homomorphisms $a_1,\dots,a_n,a_{n+1},\dots,a_n$ such that $a_1,\dots,a_r$ is an

orthonormal base in $[\![a_1,\dots,a_n, b_1,\dots,b_m]\!]$ [see (V.R.- (5.9)]

Then $\prod_{j=1}^{m} b_j$ is a linear combination of functions $g_1,\dots,g_l$ such that for each $i \in \{1,\dots,l\}$ there are $c_{i(1)},\dots,c_{i(m)} \in \{a_1,\dots,a_r\}$ with $g_i = \prod_{j=1}^{m} c_{i(j)}$. Assume that $q \in \{1,\dots,l\}$ is such that $\#\{c_{i(1)},\dots,c_{i(m)}\} < m$ for each $i \in \{1,\dots,q\}$ and $\#\{c_{i(1)},\dots,c_{i(m)}\} = m$ for each $i \in \{q+1,\dots,l\}$. From the fact that $a_1^2,\dots,a_r^2$ are also homomorphisms, it follows that for each $i \in \{1,\dots,q\}$, g_i is a product of at most m-1 homomorphisms. Now use the induction q times together with (6.26) to deduce that there is an orthonormal set of homomorphisms $a_1,\dots,a_k$ such that

$$g_i \in [\![\{\bar{x} * \prod_{j=1}^{k} a_j \mid x \in G\}]\!] \quad [i \in \{1,\dots,q\}].$$

By using again (6.26) we infer that the functions $g_{q+1},\dots,g_l$ are also in

$$[\![\{\bar{x} * \prod_{j=1}^{k} a_j \mid x \in G\}]\!]$$

and we are done. □

(6.28) THEOREM. Let $f \in C(G)$ be such that $[\![\{\bar{x} * f \mid x \in G\}]\!]$ is finite dimensional. Then there is an orthonormal set of homomorphisms $a_1,\dots,a_k$ such that

$$[\![\{\bar{x} * f \mid x \in G\}]\!] \subset [\![\{\bar{x} * \prod_{j=1}^{k} a_j \mid x \in G\}]\!].$$

<u>Proof</u>. Use (6.9) and (6.27). □

(6.29) THEOREM. Let $a_1,\dots,a_n$ be an orthonormal set of homomorphisms. Let $B = \{\prod_{j\in J} a_j \mid J \subset \{1,\dots,n\}\}$ and $D = [\![\{\bar{x} * \prod_{j=1}^{n} a_j \mid x \in G\}]\!]$. Let $U : G \to L(D,D)$ be the representation defined by

$$U_x f = \bar{x} * f \quad [x \in G \text{ and } f \in D].$$

Let $\underline{U}_x$ denote the matrix of U_x with respect to the orthonormal base B of D. Then

$$(\underline{U}_x)_{I,J} = \prod_{j \in J\setminus I} a_j(x) \qquad \text{if } I \subset J$$

and

$$= 0 \qquad \text{if } I \not\subset J.$$

Proof. $x * \prod_{j\in J} a_j = \Sigma\{(\prod_{j\in J\setminus I} a_j(x)) \prod_{j\in I} a_j \mid I \subset J\}$. □

Now we are able to describe the finite dimensional representations of G.

(6.30) THEOREM. Let E be a finite dimensional vector space and let $U : G \to L(E,E)$ be a continuous representation. Then there is $n \in \mathbb{N}$ together with a linear map A from E into $C(G)^n$ such that

$$AU_x e = (\bar{x} * (Ae)_1, \ldots, \bar{x} * (Ae)_n)$$

for each $e \in E$ and each $x \in G$. Let for each $i \in \{1,\ldots,n\}$, Π_i denote the projection of $(C(G))^n$ onto the i^{th}-component and let $A_i \underset{D}{=} \Pi_i \circ A$. Then for each $i \in \{1,\ldots,n\}$ there are natural numbers $1(i),\ldots,m(i)$ together with an orthonormal set $a_{1(i)},\ldots,a_{m(i)}$ of homomorphisms $G \to K$ such that

$$\mathrm{Im}(A_i) \subset [\![\{\bar{x} * \prod_{j=1(i)}^{m(i)} a_j \mid x \in G\}]\!].$$

Denote $D_i = [\![\{\bar{x} * \prod_{j=1(i)}^{m(i)} a_j \mid x \in G\}]\!]$ and $B_i = \{\prod_{j\in J} a_j \mid J \subset \{1(i),\ldots,m(i)\}\}$.

Let $V_i : G \to L(D_i,D_i)$ be the representation defined by

$$(V_i)_x(f) = \bar{x} * f \quad [x \in G,\ f \in D]$$

and let $\underline{(V_i)_x}$ denote the matrix of $(V_i)_x$ with respect to B_i.

Then

$$((\underline{V_{\underline{i}}})x)_{I,J} = \prod_{j \in J \setminus I} a_j \qquad \text{if } I \subset J$$

and

$$= 0 \qquad \text{if } I \not\subset J$$

[where $I,\ J \subset \{1(i),\ldots,m(i)\}$].

<u>Proof</u>. Use (5.19), (6.28) and (6.29). $\square$

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INDEX OF SYMBOLS

INDEX OF TERMS

SAMENVATTING

K stelt in dit proefschrift steeds een niet-Archimedisch gewaardeerd, volledig, lichaam voor. We onderstellen steeds dat de karakteristiek van het restklassenlichaam van K gelijk is aan het priemgetal p. De objecten die bestudeerd worden zijn groepalgebra's van een commutatieve, torsieve groep. [Een torsieve groep is een topologische, Hausdorff, groep, waarvoor er een omgevingsbasis van de identiteit is bestaande uit open ondergroepen en waarvoor iedere compacte verzameling is bevat in een compacte ondergroep.] Daar niet iedere compacte, torsieve groep een K-waardige Haarmaat heeft en we dus niet altijd de convolutie van twee functies kunnen definiëren, beperken we ons voornamelijk tot de groepalgebra M(G).

In hoofdstuk I wordt het, meestal bekende, materiaal verzameld dat nodig is in de hoofdstukken II-VI.

In hoofdstuk II bewijzen we dat iedere commutatieve torsieve groep G op een natuurlijke manier de (topologische) directe som is van een p-primaire groep G_1 [dit is een groep die "alles met p te maken heeft"] en een p-vrije groep G_2 [dit is een groep die "niets met p te maken heeft"].

We laten zien dat informatie over de groepalgebra's $M(G_1)$ en $M(G_2)$ ons informatie oplevert over de groepalgebra M(G), o.a. wat betreft de homomorfismen van M(G) naar K.

Verder tonen we aan dat er twee belangrijke types van p-primaire groepen zijn, namelijk die van type Z_p en die van type Q_p, in de zin dat iedere p-primaire compacte groep een semidirect product, en vaak zelfs een direct product, is van zulke groepen.

In hoofdstuk III worden groepen van type Z_p bestudeerd. Het blijkt dat als G een directe som is van torsieve groepen G_1 en G_2, waarbij G_1

van type Z_p, dat dan $M(G)$ een Banach moduul is over $M(G_1)$ met $||\mu * \nu|| = ||\mu|| \; ||\nu||$ voor alle $\mu \in M(G_1)$ en $\nu \in M(G)$. Dit impliceert in het bijzonder dat de norm in de maatalgebra van een groep van type Z_p multiplicatief is. We laten ook zien dat de enige groepen waarvoor de norm in de maatalgebra $M(G)$ multiplicatief is juist die van type Z_p zijn.

Voor speciale groepen G van type Z_p laten we zien dat er een 1-1 correspondentie bestaat tussen de homomorfismen van $M(G)$ naar K en de continue K-waardige karakters van G naar K. We bewijzen verder nog dat een maat op een groep van type Z_p inverteerbaar is dan en slechts dan als zijn Fourier-Stieltjes getransformeerde overal ongelijk aan nul is.

In hoofdstuk IV worden p-primaire groepen bestudeerd in het geval dat de karakteristiek van K nul is.

We poneren dat iedere K-waardige idempotent op een p-primaire groep eindig support heeft. Deze bewering wordt bewezen in het geval dat G van type Z_p is en voor het geval dat $G = C_p^I$, waarbij I een index verzameling is. Verder worden nog de Fourier-Stieltjes getransformeerden van maten op p-primaire groepen bestudeerd. Analoge vragen als gesteld in hoofdstuk III worden bediscussieerd.

In hoofdstuk V worden p-primaire groepen bestudeerd in het geval dat de karakteristiek van K ongelijk aan nul is. We bewijzen dat in deze situatie de enige idempotente maten op p-primaire groepen de trivialen zijn. Er wordt aangetoond dat een maat μ op een p-primaire groep G van eindige orde inverteerbaar is dan en slechts dan als $\mu(G) \neq 0$. Dit houdt in dat voor een p-primaire groep G altijd geldt dat de constante functies in iedere niet-triviale, gesloten schuifinvariante deelruimte van $C(G)$ zitten. Met behulp van dit resultaat laten we zien dat de bestudering van eindig dimensionale continue representaties van een groep van type C eigenlijk de bestudering is van eindig dimensionale schuifinvariante deelruimten van C(G).

In hoofdstuk VI worden de eindig dimensionale representaties uitgerekend van zeer speciale groepen, namelijk die isomorf met C_2^I voor een index verzameling I, in het geval dat de karakteristiek van K twee is. Om dit doel te bereiken bepalen we eerst de eindig dimensionale schuifinvariante deelruimtes van $C(C_2^I)$. Er wordt aangetoond dat voor iedere eindig-dimensionale schuifinvariante deelruimte V van $C(C_2^I)$ er additieve homomorfismen $a_1,\ldots,a_n$ zijn met

$$V \subset [\{\bar{s} * \prod_{i=1}^{n} a_i \mid s \in G\}].$$

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 14 juli 1949 geboren in Oud-Gastel. In 1967 behaalde hij het h.b.s.-B diploma aan het Thomas More College te Oudenbosch.

Vervolgens studeerde hij wis- en natuurkunde aan de Katholieke Universiteit te Nijmegen. Hij volgde colleges bij o.a. de hoogleraren J.H.de Boer, R.A.Hirschfeld, J.J.de Iongh, A.H.M.Levelt, A.C.M.van Rooij, H.A.M.J.Oedaijrajsingh Varma en H.de Vries. In 1972 legde hij het doctoraalexamen wiskunde af.

Van 1972 tot 1979 was hij als wetenschappelijk medewerker verbonden aan het Mathematisch Instituut te Nijmegen.

Thans is hij leraar wiskunde op het Merletcollege te Cuijk.

Van zijn hand verscheen ook het artikel "On Cardinal number related to the Banach Spaces $L^p(G)$ and $M(G)$".

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Stelling 1.

Zij G een lokaal compacte, commutatieve groep. We onderstellen $\#G$ oneindig. Laat n het minimale kardinaalgetal zijn waarvoor een basis $\mathcal{B}$ van de topologie op G bestaat met $\#\mathcal{B} = n$. Zij m het minimale kardinaalgetal waarvoor er een omgevingsbasis $\mathcal{O}$ van nul bestaat met $\#\mathcal{O} = m$. J_m is de collectie van oneindige kardinaalgetallen kleiner dan of gelijk aan m. T is de eenheidscirkel. Dan geldt dat de Banach ruimte der Radon maten op G isometrisch isomorf is met de Banach ruimte

$$l^1(n.2^m) \oplus (n.2\ ).L^1(T^s).\ s \in J_m.$$

Stelling 2.

Laat $1 \leq p \leq \infty$. Voor G en H lokaal compacte, compact voortgebrachte commutatieve groepen zijn de volgende uitspraken equivalent:

(i) de Banach ruimten M(G) en M(H) zijn isometrisch isomorf,

(ii) de Banach ruimten $L^p(G)$ en $L^p(H)$ zijn isometrisch isomorf.

Als we de eis van compact voortgebrachtheid laten vallen, wordt (i) wel door (ii) geimpliceerd, maar niet (ii) door (i).

Stelling 3.

Voor G een lokaal compacte, commutatieve groep zijn de volgende uitspraken equivalent:

(i) ieder ideaal in L'(G) is de kern van een convolutie operator H_θ met θ bijna periodiek.

(ii) G is σ-compact.

(Zie: G.Crombez, "Ideals in $L_1(G)$ and almost periodic functions"; Quaestiones Mathematicae 3 (1978), 49-52.)

Stelling 4.

Laat G een lokaal compakte, niet compakte, commutatieve groep zijn.

Zij $I \subset L^1(G)$ een gesloten, niet triviaal met begrensde benaderende eenheid $(e_\lambda)_{\lambda\in\Lambda}$. Dan $\liminf\limits_{\lambda\in\Lambda} ||e_\lambda|| \geq 2$.

Stelling 5.

Laat H een lokaal compakte, commutatieve halfgroep zijn die (topologisch) ingebed kan worden in een lokaal compakte groep. Dan zijn equivalent:

(i) er is een (niet triviaal) ideaal I binnen $L^1(H)$ met begrensde benaderende eenheid;

(ii) $L^1(H)$ heeft een begrensde benaderende eenheid.

Stelling 6.

Laat $\mathcal{DB}_1$ de klasse van doorlopende funkties zijn op een interval I die behoren tot de eerste Baireklasse. Voor $f \in \mathcal{DB}_1$ zijn dan equivalent:

(i) $fg \in \mathcal{DB}_1$ voor alle $g \in \mathcal{DB}_1$,

(ii) f voldoet aan de volgende eisen:

(a) als x discontinuiteitspunt van f is, dan $f(x) = 0$;

(b) als x discontinuiteitspunt van rechts is, dan is er rij $x_n \downarrow x$ met $f(x_n) = 0$;

(c) als x discontinuiteitspunt van links is, dan is er rij $x_n \uparrow x$ met $f(x_n) = 0$.

Stelling 7.

Voor een niet-Archimedisch gewaardeerd lichaam K met karakteristiek van het restklasselichaam ongelijk aan nul, zijn de volgende uitspraken equivalent:

(i) K is sferisch volledig,

(ii) laat $V, W \subset K$ polynoom convex zijn, zodanig dat $V \cap W = \emptyset$.

Dan $P(V \cup W) \cong P(V) \times P(W)$.

(Zie: M.van der Put, Non-Archimedean function algebras, Indag. Math., 33 (1971), 60-77.)

Stelling 8.

Laat K een niet-Archimedisch gewaardeerd lichaam zijn met karakteristiek $K = p > 0$. Zij G een separabele, compacte groep waarvoor geldt dat ieder element in G orde een macht van p heeft. Laat $f \in C(G)$, $f \neq 0$, zo zijn dat f als enige verdichtspunt nul heeft. Dan

$$\mathrm{clo}\,[\{\bar{s} * f \mid s \in G\}] = C(G).$$

Stelling 9.

Zij K een niet-Archimedisch gewaardeerd lichaam met karakteristiek $K = p > 0$. Laat G een compakte, commutatieve groep zijn waarvoor geldt dat de elementen van G waarvan de orde en macht van p is, dicht liggen. Laat $G_0 \subset G$ een gesloten ondergroep zijn met $G_0 \cong Z_p$. Dan is er een continue functie α op G met

$$\alpha(0) = 1$$

$$\alpha(x+y) = \alpha(x)\alpha(y) \text{ voor alle } x \in G_0,\ y \in G.$$

Stelling 10.

Laat K een niet-Archimedisch gewaardeerd lichaam zijn met karakteristiek $K = p > 0$. Zij G een compakte commutatieve groep waarvoor geldt dat de elementen waarvan de orde een macht van p is, dicht liggen. Dan bestaan er continue reducibele representaties van G die geen irreducibele deel-representaties van G bevatten.

Stelling 11.

Als stoplichten in Nederland minder lang op rood zouden staan, zou de verkoop van sigaretten toenemen.